Ambiguity, Trust, and the Family

Christian Wurm
Universität Düsseldorf
cwurm@phil.hhu.de

August 26, 2022

Contents

Abstract

We investigate propositional logics which enrich classical logic with a binary connective '‖' representing ambiguity. Some of these logics have been established in the literature. We briefly present and compare the most interesting of these. We generalize all existing approaches by defining the family of ambiguity logics by certain basic requirements they have to meet. We introduce further examples of ambiguity logics, investigate the structure of the family, and show how conceptual properties correspond to formal properties. The most important notion for ambiguity logics is the one of trust: every reasonable ambiguity logic is either distrustful or trustful, every distrustful logic can be extended to a trustful logic, but never vice versa. Reasoning with ambiguity necessarily gives rise to logical pluralism: distrustful logics are closed under substitution of equivalents, trustful logics are closed under uniform substitution – but closure under both results in triviality.

1 Introduction

Intro and short summary We say a word/phrase/proposition is ambiguous, if it has two or more clearly distinct meanings, only one of which is usually intended in a single usage. In traditional philosophical literature, ambiguity is considered as detrimental and hostile to sound reasoning, and one of the main tenets of formal logic is to avoid ambiguity. However, as a matter of fact, in natural language, ambiguity is uniquituous, and humans deal with this very successfully and efficiently. This has lead to the endeavor to formalize reasoning with ambiguity by means of logics which allow to explicitly represent ambiguity with an additional binary connective '‖'. Of course, '‖' has to satisfy a number of combinatorial and inferential properties in order to qualify as a representative for ambiguity. There have been several interesting and more or less independent approaches to this formalization, the most interesting for this line of work being ([16],[15],[18]).

The main result of this article is plain and simple: there exists an (interesting!) FIELD OF AMBIGUITY LOGIC, with its own characteristics and its own systematicity, which puts all these different approaches into context and perspective and allows to relate and construct new logics with a set of general formal tools and methods. Importantly, we restrict ourselves to PROPOSITIONAL AMBIGUITY LOGICS, by which we mean logics in the language of propositional logic with the additional connective '‖'. Moreover, we restrict ourselves to logics which (conservatively) extend classical logic. The most important notion for ambiguity logics is the notion of **trust**, which is central for the main cornerstones of this field: the Fundamental Theorem (presented already in [18]) states that a non-trivial ambiguity logic cannot be closed both under uniform substitution and substitution of equivalents. This leads to the distinction between between **trustful logics** and **distrustful logics**: trustful logics are closed under uniform substitution of atoms, hence each inference can be seen as a generally valid scheme regardless of content, but on the other side inferentially equivalent formulas are *not* generally interchangeable in all contexts. Distrustful logics have exactly the opposite properties: inferential equivalence entails exchangeability in all contexts, but uniform substitutions of atoms do not preserve validity of inference (see Definition 5). The In-Out-Lemma states that every trustful logic, where at least equivalent *classical* formulas are interchangeable, contains a distrustful logic as its **inner logic** (an important notion to be made precise). Finally, the Trust Theorem states that every non-trivial distrustful logic can be extended to a non-trivial trustful ambiguity logic, but never vice versa. This is the mathematical counterpart to the intuition that in a situation of trust, we accept more valid arguments than in a situation of distrust.

In this general perspective, we present the most important existing logics TAL (trustful, with commutative version cTAL) and DAL (distrustful, with non-commutative version $\bar{\mathsf{c}}\mathsf{DAL}$), and provide some new results on them. In particular, we show that whereas TAL is a very reasonable logic of trust, cTAL is trivial. We introduce the notion of the congruence algebra of a logic, which is the Lindenbaum-Tarski algebra not based on equivalence ($\dashv\vdash$), but exchangeability (we write $\equiv$). This gives rise to the distinction between inner and outer logics: the inner logic is defined by $\leqq$, which is (provided some conditions on $\wedge, \vee$) the order defined by $\equiv, \wedge, \vee$. As we have said, if $\leqq$ contains the classical consequence $\vdash_{\mathsf{CL}}$, then it defines a distrustful ambiguity logic. Interestingly, we can show that DAL is the inner logic both of cTAL, a rather large trustful logic, and $L_{\blacklozenge\blacksquare}$, the smallest of all ambiguity logics.

But most importantly, we put the existing logics in a larger context by introducing the family of ambiguity logics. Since this notion is rather abstract and not easy to grasp, we introduce this only in section 5. We also provide first results on the structure of the family, showing that it is a complete bounded lattice. We introduce a number of additional techniques to construct new ambiguity logics, many of which have rather odd properties, but one of the main results of the article is exactly this: there is no and there cannot be an ambiguity logic without *some* odd property. The list of methods we use to construct ambiguity logics are the following: 1. Proof theory, 2. Truth operators, 3. Trust closure, 4. Algebraic extensions (extending the congruence algebra) of existing logics.

To sum up, we aim to give a a first, but more or less complete overview of the field of propositional ambiguity logics in our sense, and provide a toolbox of methods and concepts to understand these logics. We will not deal with possible applications here, but the most important point in this regard is: in treating ambiguity, logical pluralism is unavoidable.

Structure of the text Section 2 provides some background knowledge and a list of core properties of ambiguity, some of which we think every logic for ambiguity has to satisfy, some of which we think might be desirable, but do not need to be satisfied. These properties have already been presented in [18], but they are indispensable for everything which follows, so we present them in a compact, yet self-contained fashion.

In Section 3 starts with a glossary, then presents the Fundamental Theorem: every non-trivial logic of ambiguity is either not closed under uniform substitution, or not closed under substitution of equivalents. We call logics which are closed under uniform substitution **logics of trust**, and we call logics which are closed under substitution of equivalents **logics of distrust**, for reasons we will explain. We present the two most prominent examples of each kind, the distrustful logic DAL (introduced in [16], with non-commutative version $\bar{\mathsf{c}}$DAL) and the trustful logic TAL (introduced in [18], with commutative version cTAL). We will see in particular that contrary to TAL, cTAL is not reasonable, and probably trust does not seem to go well together with commutative ambiguity.

Section 4 introduces the concept of a congruence algebra of a logic. In a logic closed under substitution of equivalents, this corresponds to the Lindenbaum-Tarski algebra. For trustful ambiguity logics, it does not: we construct congruence classes, the elements of the algebra, based on $\equiv$, which means *interchangeability in all contexts* $\Gamma[-]$, not $\dashv\vdash$. We will then investigate the congruence algebra of TAL, cTAL and DAL. We will also show that the pre-order defined by substitutability in all contexts, denoted by $\leqq$, has an interesting property: in the case of reasonable (classically congruent) trustful logics, $\leqq$ defines a distrustful ambiguity logic, which we call the **inner logic**. Even more surprisingly we show that DAL is the inner logic of cTAL.

Section 5 formally defines the family of ambiguity logics. We have a look at its lattice structure and some other basic properties. We present other methods for constructing ambiguity logics: one method is actually old, by means of truth operators (introduced in [15]). Next we introduce the concept of trust closure, and present the Trust Theorem, which both relate distrustful and trustful logics. We also present the minimal ambiguity logic, and the minimal trustful ambiguity logic. Since logics resulting from trust closure seem to have rather odd properties, we introduce the notion of algebraic extension, where we enlarge an (outer) logic by adding new equivalences to its inner logic.

Section 6 firstly gives a list of prominent open questions and problems, and then a short summary of the main results and, most importantly, the lesson to be learned.

2 The Notion of Ambiguity

2.1 Some Background and Prejudices on Ambiguity

The main field where ambiguity arises is in natural language, and in fact ambiguity is ubiquitous there. We will hardly find an utterance which is not ambiguous by the most rigid standards.[1] Works on disambiguation in Computational Linguistics fill books, and works on ambiguity in Linguistics do so too. Nonetheless, there are many very commonly shared prejudices about ambiguity, and even though they have been falsified many times, they maintain some vitality.

[1] For example, operationalizing "reading" as SynSet in WordNet.

We provide a short list, were we quickly explain why each is wrong with some further reading.

Prejudice 1: Humans always disambiguate It is true that humans often effortlessly disambiguate utterances, up to the point of not even noticing ambiguity. But it is not true that they *always* do this, and it is neither true that disambiguations are always unique. Rather than that, many utterances remain ambiguous, but humans do not seem to have any trouble with understanding and reasoning nonetheless (see for example [5]). An easy illustration is given by inter-annotator agreement for anaphora resolution or word sense disambiguation: it is rarely above 90%, whereas the annotated text does usually not pose any problem to understanding (see also [15], [16] for excellent expositions). An ambiguous utterance as

(1) `The first thing that strikes a stranger in New York is a big car`

obviously entails that in New York there are (rather many) big cars. Hence one fundamental insight underlying this work (and the whole topic) is that utterances often remain ambiguous, but this does not pose any serious problems for understanding and reasoning. Humans can perfectly reason with ambiguous utterances. Disambiguation, or at least semantic disambiguation, on the other hand, is simply one special case of reasoning with ambiguity, as we will see below.

Prejudice 2: Ambiguity is syntactic in nature This comes in many flavors: for example, some people sustain there is no ambiguous word `plant`, but two words $\texttt{plant}_1$, $\texttt{plant}_2$, which happen to look and sound the same (see [12] for arguments against this). But also Montague's approach to quantifier scope ambiguity, as in

(2) `Every boy loves a movie`

is an example: there are two unambiguous semantic representations, depending on which (semi-)syntactic rules we use to construct the meaning. The trick is: the translation from form to meaning IS NOT FUNCTIONAL (or deterministic): there are several ways to translate, and this way ambiguity never enters into semantics. Apart from the fact that this comes with problems of its own, one main problem is: in this case we cannot reason with ambiguity. We have to choose a single reading of a sentence in a more or less arbitrary fashion, or based on some local information, and then proceed with this. Sound (and complete) reasoning with ambiguity however presupposes that we have all readings available in some semantic representation, and this will often be necessary to infer the most plausible reading. This topic is discussed in much more detail in both [18] and [16].

Prejudice 3: Ambiguity is disjunction "Outsiders" tend to treat ambiguity simply as the DISJUNCTION OF MEANINGS (see [12] for an overview and counterarguments). However, it has been observed very early on that this is necessarily inadequate (see for example [9]): if we take an ambiguity $\alpha\|\beta$, where $\alpha \models \beta$, then $\alpha \vee \beta$ is obviously equivalent to β, and there would not even exist an ambiguity in any reasonable sense. Now this is exactly what happens in quantifier scope ambiguity, as in (2). A number of other problems arise: disjunction satisfies DeMorgan laws, ambiguity satisfies Universal Distribution (see below for explanation). Hence ambiguity is *not* disjunction.

Prejudice 4: Ambiguity can be treated by a meta-representations A very common approach for representing ambiguity (as e.g. in the quantifier case) is to use a sort of META-SEMANTICS, whose expressions *underspecify* logical representations (see [11],[4] for overview). Famous cases in point would be Cooper storage and Hole Semantics. The problem is: the meta-language does not really constitute a satisfying semantic representation, unless it is itself a logic with a well-defined inference and meaning. If ambiguity becomes "underspecification", we still have to define the valid inferences and denotational properties of the underspecification language, otherwise we do not really meet the basic tenets of a semantics. On the other hand, if we start investigating this, we are back to the endeavor we are approaching here, so there is nothing we have gained. So, this approach is not really "wrong" – it simply does not address the underlying problem.

Those were of course only the most prominent of prejudices. Refuting them leads to what we can call the **fundamental assumption**, namely that humans can reason with ambiguity, and that ambiguity is a semantic relation between propositions.

2.2 Key Aspects of Ambiguity

We investigate propositional logics with a binary ambiguity connector '$\|$' satisfying some key properties. We will now give the list of these properties of ambiguity; they are separated into 1. MANDATORY PROPERTIES, which have to be satisfied, and 2. FACULTATIVE PROPERTIES, which might be desirable but are not mandatory. We have to make one important distinction from the outset, namely the one between what we call LOCAL and GLOBAL ambiguity. For example the word `can` is ambiguous between a noun and an auxiliary; however, it will probably not contribute to the ambiguity of any sentence, because the correct syntactic category can always be inferred from its morpho-syntactic context, and hence the ambiguity never really enters into semantics. This sort of local ambiguity is of no interest here. What is interesting for us is **global ambiguity**, which cannot be disambiguated on the base of morpho-syntactic combinatorics. Note that this distinction is based on whether ambiguity properly enters semantics; whether we can disambiguate with a high plausibility based on semantic context or not is of no relevance! This whole work is about ambiguity in semantics, and therefore we are only interested in ambiguity at the sentence level, since sentences (sloppily) correspond to logical propositions, and ambiguity for us is a relation between propositions. Here comes the list of properties for '$\|$':

Mandatory: Discreteness This is one main intuitive feature of ambiguity, in particular distinguishing it from vagueness: in ambiguity, we have a finite (usually rather small) list of meanings between which an expression is ambiguous, and those are clearly distinct.

Mandatory: Universal Distribution For the combinatorics of '$\|$', the most prominent, though only recently focused (see [19]) feature of ambiguity is the fact that it equally distributes over all other connectives. To see this, consider the following examples:

(3) a. `There is a bank.`
    b. `There is no bank.`

(3-a) is ambiguous between $m_1 =$"there is a financial institute" and $m_2 =$"there is a strip of land along a river". When we negate this, the ambiguity remains, with the negated content: (3-b) is ambiguous between $n_1 =$"there is no financial institute" and $n_2 =$"there is no strip of land along a river", and importantly, the relation between the two meanings n_1 and n_2 is intuitively exactly the same as the one between m_1 and m_2. This generalizes to arbitrary connectives (see [18] for longer discussion). Hence with propositional reasoning, universal distribution means

(UD)
$$\begin{array}{lll} \neg(\alpha\|\beta) & \text{is equivalent to} & \neg\alpha\|\neg\beta \\ (\alpha\|\beta)\vee\gamma & \text{is equivalent to} & (\alpha\vee\gamma)\|(\beta\vee\gamma) \\ (\alpha\|\beta)\wedge\gamma & \text{is equivalent to} & (\alpha\wedge\gamma)\|(\beta\wedge\gamma) \end{array}$$

Note that we do not use a particular equation symbol. We will use the label (UD) to refer to this type of equivalence in both algebra and in logic; same below!

Mandatory: Unambiguous Entailments An ambiguity $\alpha\|\beta$ is generally characterized by the fact that a speaker intends one of α or β. The point is: we do not know which one of the two, as for example in

(4) `Give me the dough!`

(where `dough` might mean "pastry" or "money"). From this simple fact, we can already deduce that for arbitrary formulas α, β,

(UE) $\alpha\wedge\beta$ entails $\alpha\|\beta$ entails $\alpha\vee\beta$

If we hand over both pastry and money, we have satisfied (4), and if we have satisfied (4), we must have handed one of the two. But: we cannot reduce $\alpha\|\beta$ to neither α nor β. Note that this also illustrates how ambiguity behaves rather differently from disjunction:

(5) `Give me the pastry or give me the money!`

Uttering (5), the speaker should be satisfied with an arbitrary one of the two, uttering (4), we cannot guarantee this.

Mandatory: Idempotence Obviously, an ambiguity between m_1 and m_1 is equivalent to m_1

(id) $\alpha\|\alpha$ is equivalent to α

This goes without comment. Note that if $\alpha \vee \alpha$ is equivalent to α is equivalent to $\alpha \wedge \alpha$, then this follows from (UE), but this need not hold in all ambiguity logics.

Mandatory: Conservative extension It should be clear that a logical calculus of ambiguity should be a conservative extension of the classical calculus,[2] meaning that for formulas/sequents not involving ambiguity, exactly the same consequences should be valid as before. Conversely, this means even if we include ambiguous propositions, unambiguous propositions should behave as they used to before – if there are new entailments, they should only concern ambiguous propositions. The algebraic notion corresponding to the fragment in logic is the one of a reduct, hence the notion makes equally sense in an algebraic setting.

Mandatory: Associativity This property states that given an ambiguity between more than two meanings, their grouping is irrelevant:

($\|$assoc) $\alpha\|(\beta\|\gamma)$ is equivalent to $(\alpha\|\beta)\|\gamma$

This seems natural to us, and there seems to be nothing to object to it.

Facultative: Commutativity This property states in ambiguous propositions, the order of readings does not play a role, hence

($\|$comm) $\alpha\|\beta$ is equivalent to $\beta\|\alpha$

This is an arguable property: on the one hand, the meanings of an utterance can often be ordered in primary, secondary etc. On the other hand, we often cannot tell a natural ordering, and even if it is existent, we might want to disregard it, being more concerned with absolute truth than with plausibility of a reading or likelihood of truth. From a mathematical point of view, this property will make a big difference, as we will see in many examples (and the Fundamental Theorem). We will consider both commutative and non-commutative logics.

[2]Unless of course we assume that non-classical logics are more adequate to our purposes, but we will not consider this possibility here.

Facultative: Monotonicity This is a very weak and basic property, but we prefer to make it optional to include some extreme examples of ambiguity logics. Monotonicity means that every utterance should entail itself, and this property should be preserved over *weakening* modifications: `bank` entails `bank or restaurant`, or dually `plants and animals` entails `plants`. Formally,

($\|$mon) If α entails α', β entails β', then $\alpha\|\beta$ entails $\alpha'\|\beta'$

To see why this might be arguable: in `plants and animals`, the context might be thought of as disambiguating, whereas in `plants` there is no context. Also, a very distrustful agent might say that $\alpha\|\beta$ does not entail $\alpha\|\beta$, since the entailment is not true on any reading of both. We will see $L_{\blacklozenge\blacksquare}$ as an example for this. We will also see $\tau L_{\blacklozenge\blacksquare}$, in which every formula entails itself, but it does not generally allow for weakening. Importantly, ($\|$mon) does not require that ambiguous terms are used consistently in one sense, it is strictly weaker (see DAL as an example).

Facultative: Uniform Usage The property of (UU) is the conceptual counterpart to our (mathematical) distinction between trustful logics (as $\mathsf{TAL}, \mathsf{cTAL}$) and distrustful logics (as $\mathsf{DAL}, \bar{\mathsf{c}}\mathsf{DAL}$). We formulate it as follows:

(UU) In a given context, (globally) ambiguous terms must be used consistently in *only one* sense.

There are good reasons for assuming this: Imagine someone telling you something about `plants`, and you struggle to understand what he is trying to tell. Now, for your interpretation it makes a huge difference whether you can reasonably assume that the term `plant` in the entire discourse is used consistently in one sense, or not. In the former case, you can try to disambiguate the term globally and from there make as much sense as possible. In the latter case, for each utterance, you have to take both readings into account. The classic work by [20] gives evidence for consistent usage in texts.

On the other hand, there are good reasons for refuting it: a formula of the form $\alpha \vee \neg\alpha$, with ambiguous α, need not be necessarily true in a very simple and reasonable interpretation of ambiguity. Dually, a sentence like

(6) `He is dead and he is not dead.`

is not necessarily a contradiction, `dead` could be used in two different senses, say medical and spiritual.[3] Apart from this, the notion of "context" remains vague, and we *can* use the same word with different meanings in the same sentence, as in `I spring over a spring in spring.`[4]

There is no (semi-)(in)equality which corresponds to (UU). (UU) rather corresponds to **closure under uniform substitution**: whatever ambiguous

[3] See [16], who take this for granted.

[4] Thanks to an anonymous reviewer for this example; obviously, a lot of ambiguity remains local in this example.

term we use (as a substitute of an atomic proposition), we use it in *one sense*. In the distrustful approach (without (UU)), classical theorems are no longer valid if constituted by ambiguous propositions, and classical inferences (like Modus Ponens) usually fail if applied to ambiguous propositions, whereas in trustful reasoning, they remain valid. We will explain this in more detail below (see also [18] for further discussion).

Facultative: Law of Disambiguation We have said that disambiguation is only a special case of reasoning with ambiguity. We state the Law of Disambiguation as follows:

(LoD) $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$

(here we subsume the case where α or γ is empty). Hence ambiguous formulas can be disambiguated to less ambiguous/unambiguous formulas. We will see that this law is not entirely unproblematic: it follows from ($\|$mon) and (UD), but only if we satisfy Uniform Usage (UU), hence it is satisfied in most logics of trust. For distrustful logics, LoD normally only holds for classical (unambiguous) β:

(wLoD) $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$, provided β is unambiguous

Hence we have a weak LoD (unambiguous β), and a general LoD (arbitrary β). We will see the (very distrustful) logic $L_{\blacklozenge\blacksquare}$, which does not satisfy ($\|$mon), and hence does not even satisfy (wLoD).

This is the list of key properties for ambiguity. Let us start the logical investigations.

3 2×2 Logics of Ambiguity: Trust and Order

3.1 A Glossary of Conventions

In this section, we will introduce 2×2 logics of ambiguity, according to the two features ($\pm$)trust and ($\pm$)$\|$-commutativity. This subsection covers most of the notational conventions. It can be skipped at first reading, and consulted if the reader is unsure over the meaning of some formalism.

We denote the classical sequent calculus by CL. All logics we consider here will be in the same formula language, which (conservatively) extends classical logic with an additional binary connective '$\|$'.[5] More formally: we have a set *Var* of propositional variables, and define the set of well-formed formulas $\mathit{Form}(\mathsf{AL})$ by

1. if $p \in \mathit{Var}$, then $p \in \mathit{Form}(\mathsf{AL})$;
2. if $\phi, \chi \in \mathit{Form}(\mathsf{AL})$, then $(\phi \wedge \chi), (\phi \vee \chi), (\neg\phi) \in \mathit{Form}(\mathsf{AL})$;

[5] Our approach differs from the usual approach to substructural logic in that we *extend* classical logic with a substructural connective, whereas usually, one considers logics which are proper *fragments* of classical logic (see [7],[10]).

3. if $\phi, \chi \in Form(\mathsf{AL})$, $(\phi \| \chi) \in Form(\mathsf{AL})$;
4. nothing else is in $Form(\mathsf{AL})$.

If we can derive a formula without 3., we say it is in $Form(\mathsf{CL})$, or simply classical. As usual, we will omit outermost parentheses of formulas. We will generally use lowercase Greek variables $\alpha, \beta, \gamma, ...$ for arbitrary formulas, $p, q, r, ...$ for propositional variables. Moreover, we will sometimes use $a, b, c, ...$ as variables for classical formulas (in $Form(\mathsf{CL})$). Hence $\alpha = a_1 \| a_2$ means that a_1, a_2 are classical formulas.

For proof theory, we need to generalize classical contexts and sequents. We do this by distinguishing two different types of contexts: The classical context is denoted by (...**,** ...), which basically embeds classical logic (and allows to introduce classical connectives ($\wedge$ on the left, $\vee$ on the right, $\neg$ by side change). The ambiguous context is denoted by (...**;** ...), which allows to introduce the new connective '$\|$'. Different contexts can be arbitrarily embedded within each other.[6] We call the resulting structures **multi-contexts**, and denote them by uppercase letters (mostly Greek, but sometimes also Latin). For given multi-contexts Δ, Γ, we call a pair $\Delta \vdash \Gamma$ a **multi-sequent**. The calculus accordingly can be called a **multi-sequent calculus**. Hence $(\alpha\mathbf{,}\ \beta)$ is a well-formed (classical) context, $(\alpha\mathbf{;}\ \beta)$ is a well-formed (ambiguous) context. Formally:

1. ϵ, the empty sequence, is a well-formed context, which we also call the **empty context**.
2. If $\gamma \in Form(\mathsf{AL})$, then γ is a well-formed context.
3. If $\Gamma_1, ..., \Gamma_i$ are well-formed contexts, then $(\Gamma_1\mathbf{,}\ ...\mathbf{,}\ \Gamma_i)$ is a well-formed, classical context.
4. If Γ_1, Γ_2 are well-formed, non-empty contexts, then $(\Gamma_1\mathbf{;}\ \Gamma_2)$ is a well-formed ambiguous context.

Note that **;** is strictly binary. This choice is somewhat arbitrary, but seems to be the most elegant way to prevent some technical problems. **,** has no restriction in this sense. $\Gamma \vdash \Delta$ is a **well-formed multi-sequent** if both Γ, Δ are well-formed contexts. We write $\Gamma[\alpha]$ to refer to a subformula α of a context Γ; same for $\Gamma[\Delta]$, where Δ is a context. More formally, $\Gamma[-]$ can be thought of as a function from contexts to contexts. These context functions are inductively defined by

1. $[-] : \Delta \mapsto \Delta$ is a context function (the identity function).
2. If $\Gamma[-]$ is a context function, Θ_1, Θ_2 are contexts, then $((\Theta_1\mathbf{,}\ \Gamma[-], \Theta_2))$ is a context function, where $((\Theta_1\mathbf{,}\ \Gamma[-]\mathbf{,}\ \Theta_2))(\Delta) = (\Theta_1\mathbf{,}\ \Gamma[\Delta]\mathbf{,}\ \Theta_2)$.

[6] We have found this idea briefly mentioned as a way to approach substructural logic in [10], and structures similar to multi-contexts are found in [3]. They are also used in the context of linear logic, see for example [2].

3. If $\Gamma[-]$ is a context function, Θ is a context, then $(\Gamma[-]\mathbf{;}\ \Theta)$ is a context function, where $(\Gamma[-]\mathbf{;}\ \Theta)(\Delta) = (\Gamma[\Delta]\mathbf{;}\ \Theta)$. Parallel for $(\Theta\mathbf{;}\ \Gamma[-])$.

4. Nothing else is a context function.

We have a number of **conventions for multi-sequents**, to increase readability. These are important, as we make full use of them already in presenting the calculus.

- We omit the outermost brackets in multi-contexts, which we simply call contexts.

- ϵ is a neutral element for both $\mathbf{,}$, $\mathbf{;}$: $(\Gamma\mathbf{,}\ \epsilon) = \Gamma = (\Gamma\mathbf{;}\ \epsilon)$ etc.

- If $i > 2$, then $(\Gamma_1\mathbf{;}\ ...\mathbf{;}\ \Gamma_i)$ is an abbreviation both for $(\Gamma_1\mathbf{;}\ (\Gamma_2\mathbf{;}\ ...\mathbf{;}\ \Gamma_i))$ and $((\Gamma_1\mathbf{;}\ \Gamma_2\mathbf{;}\ ...)\mathbf{;}\ \Gamma_i)$ (meaning that we can use an arbitrary one of them). This abbreviation is unproblematic due to rules ensuring associativity of bracketing. If $i = 1$, it is an abbreviation for Γ_1. If $i = 0$, then it is an abbreviation for ϵ (the empty context). The latter two conventions are useful to formulate rules with more generality.

- We let $(\Delta_1\mathbf{,}\ (\Delta_2\mathbf{,}\ ...\mathbf{,}\ \Delta_i))$ and $((\Delta_1\mathbf{,}\ ...\mathbf{,}\ \Delta_{i-1})\mathbf{,}\ \Delta_i)$ just be an alternative notation for $(\Delta_1\mathbf{,}\ ...\mathbf{,}\ \Delta_i)$. Hence classical contexts do not really embed into each other. This again will allow to formulate rules in greater generality.

Another convention is the following: a logic is a pair $L = (Form, \vdash_L)$, with *Form* its set of well-formed formulas and $\vdash_L \subseteq Form \times Form$ its consequence relation. We write $\alpha \vdash_L \beta$ if the sequent is valid in L. However, sometimes it will be convenient to speak of a sequent $\alpha \vdash \beta$ independently of a logic. Then we also write $\Vdash_L \alpha \vdash \beta$ or $\nVdash_L \alpha \vdash \beta$, meaning that the sequent is (or not) valid in L.[7]

Having said this, the relation $\vdash_L$ is *always* extended from formulas to multi-contexts (also in $\mathsf{DAL}, L_{\blacklozenge\blacksquare}$ and others below). Take the functions l, r, which are simple maps from contexts to formulas:

$$l(\alpha) = \alpha = r(\alpha) \qquad r(\Gamma\mathbf{,}\ \Delta) = r(\Gamma) \vee r(\Delta)$$
$$l(\Gamma\mathbf{;}\ \Delta) = l(\Gamma)\|l(\Delta) = r(\Gamma\mathbf{;}\ \Delta) \qquad l(\Gamma\mathbf{,}\ \Delta) = l(\Gamma) \wedge l(\Delta)$$

For every logic L, we will always implicitly assume that

(7) $\Gamma \vdash_L \Delta$ iff $l(\Gamma) \vdash_L r(\Delta)$

This makes sure that $\leqq_L, \equiv_L$ (see Definition 7) are well-defined for every logic L. An important notion is the one of **ambiguous normal form**, which results from distributing out all ambiguity, until we remain with a formula $a_1\|...\|a_i$, where $a_1, ..., a_i$ are classical. (UD) ensures that every formula is equivalent to all of its ambiguous normal forms.

[7] This is usual in algebra, where both $t =_{\mathbf{BA}} t'$ and $\mathbf{BA} \models t = t'$ are synonym.

Definition 1 *We define* $anf(\phi)$ *syntactically by*

- *If* $\phi \in Form(\mathsf{CL})$*, then* $anf(\phi) = \{\phi\}$.
- $\psi \in anf(\neg\phi)$ *if and only if* $\psi = (\neg\alpha_1)\|...\|(\neg\alpha_i)$ *for some* $\alpha_1\|...\|\alpha_i \in anf(\phi)$.
- *For* $\star \in \{\wedge, \vee\}$, $\psi \in anf(\phi \star \chi)$ *if and only if* $\psi = \gamma_1\|...\|\gamma_i$*, where either (i) for some* $\alpha_1\|...\|\alpha_i \in anf(\phi)$ *and all* $j \in \{1, ..., i\}$*, we have* $\gamma_j \in anf(\alpha_j \star \chi)$*, or (ii) for some* $\alpha_1\|...\|\alpha_i \in anf(\chi)$ *and all* $j \in \{1, ..., i\}$*, we have* $\gamma_j \in anf(\phi \star \alpha_j)$.
- $\psi \in anf(\phi\|\chi)$ *if and only if* $\psi = \alpha\|\beta$*, where* $\alpha \in anf(\phi)$*,* $\beta \in anf(\chi)$.

3.2 The Fundamental Theorem

We now present the Fundamental Theorem in two versions, which (we believe) is the most important result on ambiguity logics. It entails that every non-trivial ambiguity logic lacks at least one basic closure property of abstract logics in the sense of Tarski [13]: either closure under uniform substitution, or closure under substitution of equivalents (or more strictly, transitivity).

The proof of the Fundamental Theorem can be found in [18], following up to [19],[17]. We say an algebra is of the **signature of ambiguous algebras**, if it has the form $(A, \wedge, \vee, \sim, \|, 0, 1)$, where $(A, \wedge, \vee, \sim, 0, 1)$ is a Boolean algebra and '$\|$' is a binary connective. The notion "is equivalent to" above becomes $=$ in algebra, $\dashv\vdash$ in logic, "entails" becomes $\leq$ in algebra, $\vdash$ in logic.

Theorem 2 *(Fundamental Theorem, algebraic version)*

1. *Let* $\boldsymbol{A}$ *be an algebra of the signature of ambiguous algebras which satisfies (the algebraic counterpart of) (UD),($\|$assoc),(UE),($\|$comm). Then* $\boldsymbol{A}$ *is trivial, i.e. has at most one element.*
2. *Let* $\boldsymbol{A}$ *be an algebra of the signature of ambiguous algebras which satisfies (the algebraic counterpart of) (UD),($\|$assoc),(UE). Then for all* $a, b, c \in A$*,* $\boldsymbol{A}$ *satisfies* $a\|c\|b = a\|b$

Theorem 2.1 is of course the algebraic counterpart to logical inconsistency. Theorem 2.2 shows that these algebras have strongly counterintuitive properties already without ($\|$comm), so adding commutativity makes them one-element. We quickly translate these algebraic results into results on ambiguity logics.

Corollary 3 *(Fundamental Theorem, logical version)*

1. *Let* $\mathcal{L} = (Form(\mathsf{AL}), \vdash)$ *be a logic which conservatively extends classical logic and satisfies (the logical counterpart of) (UD),($\|$assoc),(UE),($\|$comm), is closed under uniform substitution of atoms and admits the rule (cut). Then* $\mathcal{L}$ *is inconsistent.*

2. *Let $\mathcal{L} = (Form(\mathsf{AL}), \vdash)$ be a logic which conservatively extends classical logic and satisfies (the logical counterpart of) (UD),($\|$assoc),(UE), is closed under uniform substitution of atoms and admits the rule (cut). Then for all $\alpha, \beta, \gamma \in Form(\mathsf{AL})$, we have $\alpha\|\gamma\|\beta \dashv\vdash \alpha\|\beta$.*

This is a key result, because it shows that algebra and "normal" logics, closed under (cut) and uniform substitution, are **fundamentally inappropriate** for reasoning with ambiguity. The property of Corollary 3.2 is slightly weaker than inconsistency, yet it is strong enough to exclude any logic which satisfies it as a reasonable ambiguity logic. We now introduce a technical notion of triviality of a logic, which allows to uniquely refer to this property.

Definition 4 *An ambiguity logic L is **margin-trivial** (or **m-trivial**), if for all formulas $\alpha, \beta, \beta', \gamma$, $\alpha\|\beta\|\gamma \dashv\vdash_L \alpha\|\beta'\|\gamma$.*

M-triviality serves as an important benchmark to decide when an ambiguity logic is "too permissive" to be actually useful. The immediate consequence of the Fundamental Theorem is the following:

Observation 1 *The only way to incorporate the basic features of ambiguity into a reasonable system for reasoning with ambiguity is to abandon one key feature of algebra itself.*

However, there are two key features which can be abandoned while still satisfying the mandatory properties of ambiguity:

1. Uniform substitution of atomic propositions by arbitrary formulas preserves the truth of sequents: $\alpha \vdash_L \beta$, $\sigma : Var \to Form(\mathsf{AL})$ a uniform substitution, entail $\sigma(\alpha) \vdash_L \sigma(\beta)$. We call this **closure under u-substitution**.

2. Substitution of arbitrary $\dashv\vdash_L$ equivalent formulas preserves the truth of sequents: $\alpha \dashv\vdash_L \beta$ and $\Gamma[\alpha] \vdash_L \Delta$ entail $\Gamma[\beta] \vdash_L \Delta$; same on the right. We call this **closure under e-substitution**.

Let us illustrate this with two examples.

Example 1 $p \vee \neg p$ is obviously a theorem in every ambiguity logic. However, the uniform substitution $\sigma : p \mapsto p\|q$ need not preserve this: $(p\|q) \vee \neg(p\|q)$ need not be a theorem (see DAL).

Example 2 Assume $(p\|q) \vee \neg(p\|q)$ *is* a theorem in a logic L. Hence

(8) $(p \vee \neg p)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q)$

is also a theorem (just apply laws of universal distribution). However,

(9) $(r \vee \neg r)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q)$

need *not* be a theorem, even though $p \vee \neg p \dashv\vdash_L r \vee \neg r$ (conservative extension).

This is the most important and basic distinction in the study of ambiguity logics, and the most important insight is to understand that one of the two properties has to be abandoned, if we do not want to end up with m-triviality. This leaves us with **four kinds** of ambiguity logics (i.e. classical logics with '$\|$', where '$\|$' satisfies the mandatory properties of ambiguity, see Definition 49 below):

1. Ambiguity logics which are closed under e- and u-substitution. These logics are all trivial and of no interest!

2. Ambiguity logics which are not closed under e-substitution, but closed under u-substitution. We call these the (non-trivial) **logics of trust**. The reason for this is, in a nutshell, that every inference is valid as a *scheme*, regardless of the exact content of propositional variables (the "content" of a variable can be thought of as the result of the substitution).

3. Ambiguity logics which are closed under e-substitution, but not under u-substitution. We call these logics (non-trivial) **logics of distrust**. The reason is that inferences are not schemes: if we (uniformly) substitute an atom p in a formula with an ambiguous formula, a valid argument might become invalid.

4. Ambiguity logics which are closed neither under e- nor u-substitution. We cannot say these logics are generally uninteresting, but for now we have neither a motivation nor a relevant example for these logics, so we will not look at them here.

Hence we will investigate logics of kind 2. and 3., logics of trust and logics of distrust. Note that there is an interesting correlation between (necessary) formal and conceptual properties of logics: trust is closure under u-substitution ("arguments are schemes"), and the lack thereof is partly compensated by closure under e-substitution. Formal definitions are as follows (note that this definition slightly differs from the above considerations, for technical reasons):

Definition 5 *Assume $L = (Form(\mathsf{AL}), \vdash_L)$ is a logic (of ambiguity).*

1. *We say L is a **trustful logic**, if for every uniform substitution $\sigma : Var \to Form(\mathsf{AL})$, $\Gamma \vdash_L \Delta$ entails $\sigma(\Gamma) \vdash_L \sigma(\Delta)$*

2. *We say L is a **distrustful logic**, if $\Gamma[\alpha] \vdash_L \Delta$, $\alpha' \vdash_L \alpha$ entail $\Gamma[\alpha'] \vdash_L \Delta$, and $\Gamma \vdash_L \Delta[\beta]$, $\beta \vdash_L \beta'$ entail $\Gamma \vdash_L \Delta[\beta']$.*

This leaves a huge number of questions: for example, one would assume that in a situation of trust, we accept more valid arguments that in a situation of mistrust (see ((6),(14)); yet we have not required that trustful logics should include distrustful logics. We will see (Trust Theorem) that every distrustful logic is contained in a trustful logic, but never vice versa; hence this and many other questions will be answered in the sequel

Note that the definition of distrustful logics is slightly stronger than interchangeability of $\dashv\vdash$-equivalents. It is equivalent in case the logic is naturally ordered (Definition 28). However, we will see some logics where this is not the case ($\tau L_{\blacklozenge\blacksquare}, \tau\mathsf{DAL}$), and here our more general definition prevents some problems.

3.3 Trustful Reasoning with Ambiguity: TAL and cTAL

We now present the proof theory of TAL and cTAL, two logic of trust, which are closed under u-substitution, but not e-substitution. We formulate the calculus in a way to make structural rules, as far as they are desired, admissible (see [8] on the topic). Arguably, some rules could be formulated in an intuitively simpler way, but at the price of not having admissible structural rules, which can be problematic for proof search. TAL has been investigated at length in [18], so we state many properties without explicit proofs. TAL also has a sound and complete semantics, which however is not very intuitive, so we do not discuss it here. Note that we slightly change the presentation with respect to [18].

$$\text{(ax)}\ \frac{}{\alpha,\ \Gamma \vdash \alpha,\ \Delta}$$

$$(\wedge\text{I})\ \frac{\Gamma[\alpha,\ \beta] \vdash \Theta}{\Gamma[\alpha \wedge \beta] \vdash \Theta} \qquad (\text{I}\wedge)\ \frac{\Gamma \vdash \Theta[\alpha] \quad \Gamma \vdash \Theta[\beta]}{\Gamma \vdash \Theta[\alpha \wedge \beta]}$$

$$(\vee\text{I})\ \frac{\Gamma[\alpha] \vdash \Theta \quad \Gamma[\beta] \vdash \Theta}{\Gamma[\alpha \vee \beta] \vdash \Theta} \qquad (\text{I}\vee)\ \frac{\Gamma \vdash \Theta[(\alpha,\ \beta)]}{\Gamma \vdash \Theta[\alpha \vee \beta]}$$

($\wedge$I) and (I$\vee$) show how $\wedge, \vee$ correspond to '$,$ ', depending on the side of $\vdash$. For negation, we have slightly generalized standard rules which ensure distribution:

$$(\neg\text{I})\ \frac{\Gamma \vdash \Delta,\ (\alpha_1;\ ...;\ \alpha_i)}{\Gamma,\ (\neg\alpha_1;\ ...;\ \neg\alpha_i) \vdash \Delta} \qquad (\text{I}\neg)\ \frac{\Gamma,\ (\alpha_1;\ ...;\ \alpha_i) \vdash \Delta}{\Gamma \vdash \Delta,\ (\neg\alpha_1;\ ...;\ \neg\alpha_i)}$$

$$(,\ \text{comm})\ \frac{\Gamma[\Psi,\ \Theta]}{\Gamma[\Theta,\ \Psi]} \qquad (,\ \text{weak})\ \frac{\Gamma[\Delta]}{\Gamma[(\Delta,\ \Psi)]} \qquad (,\ \text{contr})\ \frac{\Gamma[(\Delta,\ \Delta)]}{\Gamma[\Delta]}$$

The absence of $\vdash$ means that the rules can be equally applied on both sides of $\vdash$. The introduction rule for $;$ is basically a generalization of ($\|$mon) and (UE).[8]

$$(\text{I};\ \text{I})\ \frac{\Gamma,\ \Lambda \vdash \Delta,\ \Psi \quad \Theta,\ \Lambda \vdash \Phi,\ \Psi}{(\Gamma;\ \Theta),\ \Lambda \vdash (\Delta;\ \Phi),\ \Psi}$$

[8] The additional rules (I$\lozenge$) and ($\lozenge$I) from [18] are actually admissible, as we will see later on.

$$(\|\mathrm{I})\ \frac{\Gamma[\alpha;\ \beta] \vdash \Theta}{\Gamma[\alpha\|\beta] \vdash \Theta} \qquad (\mathrm{I}\|)\ \frac{\Gamma \vdash \Theta[\alpha;\ \beta]}{\Gamma \vdash \Theta[\alpha\|\beta]}$$

There are two structural rules in **;** -context, namely associativity, contraction.

$$(\mathbf{;}\ \text{assoc})\ \frac{\Psi[(\Gamma;\ (\Delta;\ \Theta))]}{\Psi[((\Gamma;\ \Delta);\ \Theta))]} \qquad (\mathbf{;}\ \text{contr})\ \frac{\Gamma[\alpha;\ \alpha]}{\Gamma[\alpha]}$$

Here double lines indicate that the rule works in both directions. To ensure proper distribution and invertibility without (cut), we need the interpolation rules (inter1) and (inter2):

$$(\text{inter1})\ \frac{\Gamma[(\Delta;\ \Psi),\ \Delta] \quad \Gamma[(\Delta;\ \Psi),\ \Psi]}{\Gamma[(\Delta;\ \Psi)]}$$

$$(\text{inter2})\ \frac{\Gamma[\Psi,\ (\Delta;\ \Psi;\ \Delta')] \quad \Gamma[(\Delta;\ (\beta,\ \Psi);\ \Delta')]}{\Gamma[(\Delta;\ \Psi;\ \Delta')]}$$

These rules have alternative, equivalent formulations: (distr) is equivalent to (inter1), (subst) is equivalent to (inter2). (distr) and (subst) are easier to grasp intuitively, but they have a disadvantage that (i) they are not invertible, and (ii) they pose some problems in proving admissibility of structural rules.[9] We present them, though they are not strictly part of the calculus:

$$(\text{distr})\ \frac{\Gamma[(\Delta;\ \Psi),\ \Theta_1] \quad \Gamma[(\Delta;\ \Psi),\ \Theta_2]}{\Gamma[(\Delta,\ \Theta_1);\ (\Psi,\ \Theta_2)]} \qquad (\text{subst})\ \frac{\Gamma[\Psi] \quad \Gamma[(\Delta;\ \beta;\ \Delta')]}{\Gamma[(\Delta;\ \Psi;\ \Delta')]}$$

There is another important pair of rules of inverse distribution, which are admissible (and hence not part of the calculus, see [18]), but will play a role in some proofs:

$$(\text{invDistr1})\ \frac{\Gamma[((\Delta,\ \Theta);\ \Psi)]}{\Gamma[((\Delta;\ \Psi),\ \Theta)]} \qquad (\text{invDistr2})\ \frac{\Gamma[(\Delta;\ (\Psi,\ \Theta))]}{\Gamma[((\Delta;\ \Psi),\ \Theta)]}$$

Proofs are defined inductively as usual. We call the logic of all sequents derivable by the rules so far the **trustful ambiguity logic** TAL, and denote its consequence relation with $\vdash_{\mathsf{TAL}}$. Finally, there is **;** -commutativity:

[9]This will play no role here, but in [18] most structural rules have been proved to be admissible

$$(;\ \text{comm})\ \frac{\Gamma[(\beta;\ \alpha)]}{\Gamma[(\alpha;\ \beta)]}$$

TAL with this additional rule is called cTAL. Now consider the rule (cut), not part neither of TAL nor cTAL:

$$(\text{cut})\ \frac{\Gamma[\alpha] \vdash \Delta \quad \Theta \vdash \alpha}{\Gamma[\Theta] \vdash \Delta}$$

We call the logics with (cut) $\mathsf{TAL}^{cut}, \mathsf{cTAL}^{cut}$. Basic results are the following (which are partly obvious, partly proved in [18]):

- TAL and cTAL conservatively extend classical logic,
- $\mathsf{TAL}, \mathsf{cTAL}$ are consistent
- TAL and cTAL are closed under u-substitution
- TAL and cTAL do not admit the rule (cut) (corollary of the Fundamental Theorem).
- TAL^{cut} is trivial, cTAL^{cut} is inconsistent (corollary of the Fundamental Theorem).

This can be strengthened: define the transitive closure of a logic L as usual by just making $\vdash_L$ transitive on formulas (not contexts). The following is proved in [18]:

Lemma 6 *The transitive closure of* TAL *is trivial, the transitive closure of* cTAL^{cut} *is inconsistent.*

This gives rise to an important distinction, namely between **inner logics** and **outer logics**. The relation $\vdash$ means *entailment*, but in a trustful logic L, $\alpha \vdash_L \beta$ does *not* entail that α is logically stronger than β in *all contexts*. This notion is the corresponding inner logic $\leqq_L$:

Definition 7 *Let L be a logic. We write $\alpha \leqq_L \beta$ iff $\Gamma[\beta] \vdash_L \Delta$ implies $\Gamma[\alpha] \vdash_L \Delta$ and $\Delta \vdash_L \Gamma[\alpha]$ implies $\Delta \vdash_L \Gamma[\beta]$. We call $\leqq_L$ the **inner logic** of L, and we let $\equiv_L$ denote the reflexive closure of $\leqq_L$.*

Correspondingly, we call $\vdash_L$ the **outer logic**. In a calculus which is closed under e-congruence (corresponds to admissible (cut)), $\leqq_L$ coincides with $\vdash_L$. $\equiv_{\mathsf{TAL}}$ is the relation of *congruence*, or interchangeability. It is by definition an equivalence relation. $\alpha \leqq_{\mathsf{TAL}} \beta$ can be conceived of as: "α is logically stronger than β in all contexts", and it is by definition a pre-order (reflexive,transitive), and hence a partial order up to $\equiv_{\mathsf{TAL}}$ congruence. The following is immediate:

Lemma 8 *Let L be an ambiguity logic. Then $\leqq_L \subseteq \vdash_L$ iff for all α, $\alpha \vdash_L \alpha$. Moreover, if L is trustful and non-trivial, then $\leqq_L \subsetneq \vdash_L$.*

Proof. *If* $\alpha \vdash_L \alpha$, $\alpha \leqq_L \beta$ obviously entails $\alpha \vdash_L \beta$.

Only if Contraposition: assume there is an α such that $\alpha \not\vdash_L \alpha$. $\leqq_L$ by definition is reflexive (substitutability), hence $\leqq_L \not\subseteq \vdash_L$.

Finally, assume L is trustful and non-trivial, hence it cannot be closed under (cut). $\leqq_L$ by definition is, hence the two cannot be identical. ⊣

We will later see the logic $L_{\blacklozenge\blacksquare}$ which does not satisfy the premise $\alpha \vdash_{\blacklozenge\blacksquare} \alpha$, hence this is not trivial. For the investigation of trustful (outer) logics, the inner logic $\leqq$ is absolutely crucial. Its meaning for trustful ambiguity logics can be maybe compared to the meaning of the Deduction Theorem for Hilbert calculi: it allows to get from one derivable sequent to another quickly. It is generally not straightforward to make proofs for $\leqq, \equiv$, given a sequent calculus: for example, $\vdash_L \subseteq \vdash_{L'}$ does not generally entail $\leqq_L \subseteq \leqq_{L'}$ (see the logics $L_{\blacklozenge\blacksquare}$ and $\tau\mathsf{DAL}$ later on). However, we will show an easy way to prove properties for $\leqq_{\mathsf{TAL}}, \leqq_{\mathsf{cTAL}}$ in Lemma 30. In [18], many results can be found for TAL, in particular regarding (UD),(UE) etc.

Invertibility of a rule basically means that it can also be applied "upside down", in the indirect sense that from the derivability of the conclusion, we can infer the derivability of the premises. This depends not only on the logic, but also on the formulation of rules. For example, (inter1),(inter2) are invertible, whereas the equivalent (distr),(subst) are not. Classical logic has a fully invertible calculus. In our calculus, we can achieve *almost* full invertibility:

Lemma 9 *In* TAL *and* cTAL*, all rules are invertible except for (I*; *I) and (*, *weak).*

Proof. This consists of two parts: invertibility of all rules except (I**;** I),(**,** weak) is an easy exercise (see [18]). The non-invertibility of (I**;** I) in TAL is also straightforward, together with (**;** assoc): consider a sequent $p; (q; r) \vdash (p; q); r$: this is obviously not invertible. (**,** weak) is obviously not invertible, but admissible. ⊣

A more intricate example is the sequent $p; \neg p \vdash q; \neg q$, provable in $\mathsf{TAL}, \mathsf{cTAL}$ and DAL. As we have said, TAL and cTAL are logics of trust. There is one particular pair of rules which makes the difference, namely ($\neg$I),(I$\neg$). These are the rules which ensure that $\alpha \vee \neg\alpha$ is a theorem for arbitrary α, and $\alpha \wedge \neg\alpha$ a contradiction. These are the rules which are only sound under the assumption of uniform usage, and these are the rules which do not preserve the identity of $\leqq_{\mathsf{TAL}}$ with $\vdash_{\mathsf{TAL}}$: actually, it is possible to prove that in the calculus without negation, (cut) is admissible. ($\neg$I),(I$\neg$) are also the only rules of $\mathsf{TAL}, \mathsf{cTAL}$ not sound in DAL. Hence the unrestricted negation rule is really what makes the logics $\mathsf{TAL}, \mathsf{cTAL}$ trustful, whereas DAL restricts this rule to atoms. We think that TAL is the most reasonable logic of trust. For cTAL, we cannot say it is very reasonable, for reasons we will lay out in section 3.5. Note that $\mathsf{TAL}, \mathsf{cTAL}$ satisfy a number of other properties typical for trust:

(10) $$(\alpha\|\beta\|\gamma) \wedge \neg\beta \quad \vdash_{\mathsf{TAL}} \quad \alpha\|\gamma$$

(11) $$\alpha, \ \alpha \to \beta \quad \vdash_{\mathsf{TAL}} \quad \beta$$

Hence we satisfy Modus Ponens and the general LoD for arbitrary formulas. Especially these two properties distinguish $\vdash_{\mathsf{TAL}}, \vdash_{\mathsf{cTAL}}$ from $\vdash_{\mathsf{DAL}}$. We underline that the two entailments only hold for the outer logic, not the inner logic (see Lemma 44)!

3.4 Distrustful Reasoning with Ambiguity: DAL and $\bar{\mathsf{c}}$DAL

Whereas TAL (non-commutative) is in our view the most reasonable logic for trustful reasoning with ambiguity, DAL (commutative) in our view is the most reasonable logic for distrustful reasoning, first presented in [16], though not as a distrustful ambiguity logic. Its proof theory is characterized by

1. Using all rules of cTAL except negation rules (I¬),(¬I),
2. adding (cut), and
3. adding the following restricted negation rules (we only provide them for one side and skip duals, (contra) is obviously self-dual):

$$(\text{at}\neg)\ \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p},\ p \text{ atomic} \qquad (\text{contra})\ \frac{\Gamma \vdash \Delta}{\neg\Delta \vdash \neg\Gamma} \qquad (\text{eDN})\ \frac{\Gamma \vdash \Delta, \neg\neg\alpha}{\Gamma \vdash \Delta, \alpha}$$

(Introducing double negation is admissible). Here $\neg\Gamma$ is an abbreviation for pointwise negation of all formulas in Γ. Since DAL allows (cut), (inter1),(inter2) are admissible, so we do not need them. The advantage of this presentation is that it allows to define $\bar{\mathsf{c}}$DAL, the non-commutative version of DAL: it is simply the logic DAL without the rule (; comm).

We will focus on (commutative) DAL in this article, since DAL is much more easily accessible (and better motivated) from the semantic side. Take a classical model $M \subseteq Var$, an ambiguous formula $\alpha\|\beta$, and assume for simplicity that both α and β are classical. Then we can intuitively distinguish three cases:

1. $M \models \alpha\|\beta$ – $\alpha\|\beta$ is necessarily true, since α, β are both true
2. $M \rightsquigarrow \alpha\|\beta$ (read: falsifies) – $\alpha\|\beta$ is necessarily false, since both α, β are false
3. none of the two (true under one, false under another reading)

This is the starting point for [16]. We will initially follow this presentation, but then show a presentation using operators $\blacksquare, \blacklozenge$ (introduced by [15]) which is easier to handle. Here is the double list of verifying ($\models$) and falsifying ($\rightsquigarrow$) conditions for formulas/connectives, as presented in [16].

$\models$1 $M \models p_i$ iff $p_i \in M$.

$\models$2 $M \models \alpha \wedge \beta$ iff $M \models \alpha$ and $M \models \beta$.

$\models$3 $M \models \alpha \vee \beta$ iff at least one holds, $M \models \alpha$ or $M \models \beta$.

$\models$4 $M \models \neg\alpha$ iff $M \rightsquigarrow \alpha$

$\models$6 $M \models \alpha \| \beta$ iff $M \models \alpha \wedge \beta$

$\rightsquigarrow$1 $M \rightsquigarrow p_i$ iff $p_i \notin M$

$\rightsquigarrow$2 $M \rightsquigarrow \alpha \wedge \beta$ iff at least one holds, $M \rightsquigarrow \alpha$ or $M \rightsquigarrow \beta$

$\rightsquigarrow$3 $M \rightsquigarrow \alpha \vee \beta$ iff $M \rightsquigarrow \alpha$ and $M \rightsquigarrow \beta$

$\rightsquigarrow$4 $M \rightsquigarrow \neg\alpha$ iff $M \models \alpha$

$\rightsquigarrow$6 $M \rightsquigarrow \alpha \| \beta$ iff $M \rightsquigarrow \alpha \vee \beta$

Note the central point: for verification, '$\|$' behaves like $\wedge$, for falsification, '$\|$' behaves like $\vee$. Hence it is a connective which switches the condition according to "truth modality". This ensures its intermediate position between $\wedge$ and $\vee$.

This defines two relations between models and formulas. We want to define a logic, that is, a single relation between formulas, which corresponds to ambiguous entailments. We use the symbol $\vdash_{\mathsf{DAL}}$ to denote the consequence relation of our **distrustful ambiguity logic** DAL.

- We write $\alpha \models \beta$ iff $M \models \alpha$ implies $M \models \beta$.
- We write $\alpha \rightsquigarrow \beta$ iff $M \rightsquigarrow \alpha$ implies $M \rightsquigarrow \beta$.
- We write $\models \alpha$ if for all M, $M \models \alpha$, dually for $\rightsquigarrow$.

Definition 10 *We define the relation $\vdash_{\mathsf{DAL}}$ by $\alpha \vdash_{\mathsf{DAL}} \beta$ iff both $\alpha \models \beta$ and $\beta \rightsquigarrow \alpha$. We say $\vdash_{\mathsf{DAL}} \alpha$, or α is a theorem of* DAL *if $\models \alpha$. We say $\alpha \vdash_{\mathsf{DAL}}$, or α is a contradiction of* DAL*, if $\rightsquigarrow \alpha$.*

This is the (semantic) definition of DAL in [16]: truth of α implies truth of β, and falsity of β implies falsity of α. Obviously, for classical α, β we have $\alpha \models \beta$ iff $\beta \rightsquigarrow \alpha$ (contraposition) iff $\alpha \vdash_{\mathsf{DAL}} \beta$. So $\vdash_{\mathsf{DAL}}$ is a conservative extension of $\vdash_{\mathsf{CL}}$. For ease of argument, we give an alternative presentation, using two maps $\blacksquare, \blacklozenge$ from ambiguous to classical formulas:

$$\begin{array}{rrcrr}
\blacksquare p = & p & \quad & \blacklozenge p = & p \\
\blacksquare(\alpha \wedge \beta) = & (\blacksquare\alpha) \wedge (\blacksquare\beta) & & \blacklozenge(\alpha \wedge \beta) = & (\blacklozenge\alpha) \wedge (\blacklozenge\beta) \\
\blacksquare(\alpha \vee \beta) = & (\blacksquare\alpha) \vee (\blacksquare\beta) & & \blacklozenge(\alpha \vee \beta) = & (\blacklozenge\alpha) \vee (\blacklozenge\beta) \\
\blacksquare(\neg\alpha) = & \neg(\blacklozenge\alpha) & & \blacklozenge(\neg\alpha) = & \neg(\blacksquare\alpha) \\
\blacksquare(\alpha\|\beta) = & (\blacksquare\alpha) \wedge (\blacksquare\beta) & & \blacklozenge(\alpha\|\beta) = & (\blacklozenge\alpha) \vee (\blacklozenge\beta)
\end{array}$$

Lemma 11 *For all $\alpha \in Form(\mathsf{AL})$,*

1. *$M \models \alpha$ iff $M \models \blacksquare\alpha$ (iff $M \not\rightsquigarrow \blacksquare\alpha$)*

2. $M \rightsquigarrow \alpha$ iff $M \not\models \blacklozenge\alpha$ (iff $M \rightsquigarrow \blacklozenge\alpha$)

The first bi-implication can be proved by induction on formula complexity and straightforward checking of the conditions. $M \models \blacksquare\alpha$ iff $M \not\rightsquigarrow \blacksquare\alpha$ holds in virtue of $\blacksquare\alpha$ being classical. Same for $\blacklozenge$. The following lemma is simple, but extremely useful:

Lemma 12 *$\alpha \vdash_{\mathsf{DAL}} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ and $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$.*

Proof. *If* Assume $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ and $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$. Then by Lemma 11.1 (and classical completeness), if $M \models \alpha$, then $M \models \beta$. Moreover, by contraposition (valid for classical formulas) we obtain $M \not\models \blacklozenge\beta$ entails $M \not\models \blacklozenge\alpha$, where $M \not\models \blacklozenge\gamma$ is equivalent to $M \rightsquigarrow \gamma$ (Lemma 11.2). Hence $\alpha \vdash_{\mathsf{DAL}} \beta$.

Only if Assume $\alpha \vdash_{\mathsf{DAL}} \beta$. Hence $\alpha \models \beta$, so (Lemma 11.1) $M \models \blacksquare\alpha \Leftrightarrow M \models \alpha \Rightarrow M \models \beta \Leftrightarrow M \models \blacksquare\beta$. Thus $\blacksquare\alpha \models \blacksquare\beta$. Moreover, $\beta \rightsquigarrow \alpha$, so dually $M \rightsquigarrow \blacklozenge\beta \Leftrightarrow M \rightsquigarrow \beta \Rightarrow M \rightsquigarrow \alpha \Leftrightarrow M \rightsquigarrow \blacklozenge\alpha$. Since $\blacklozenge\alpha, \blacklozenge\beta$ are classical, $\blacklozenge\beta \rightsquigarrow \blacklozenge\alpha$ iff $\blacklozenge\alpha \models \blacklozenge\beta$. ⊣

This shows how $\vdash_{\mathsf{DAL}}$ can be very neatly reduced to $\vdash_{\mathsf{CL}}$ in a simple fashion. The operators $\blacklozenge, \blacksquare$ will later on play other important roles for the analysis and construction of ambiguity logics. Note that $\vdash_{\mathsf{DAL}}$ is also defined on contexts Γ, Δ by our general conventions, where $\Gamma \vdash_{\mathsf{DAL}} \Delta$ iff $l(\Gamma) \vdash_{\mathsf{DAL}} r(\Delta)$.

Now we can present some results. Firstly, '$\|$' is obviously commutative in this logic. Secondly, the logic is obviously consistent and a conservative extension of classical logic. Thirdly, the logic obviously allows for substitution of equivalents (logically, the rule of cut); in our terms, this means it is closed under e-substitution.

Lemma 13 *$\Gamma[\beta] \vdash_{\mathsf{DAL}} \Delta$ and $\alpha \vdash_{\mathsf{DAL}} \beta$ entails $\Gamma[\alpha] \vdash_{\mathsf{DAL}} \Delta$.*

As a next result, we see the following:

Lemma 14 DAL *satisfies (UD), (mon) and (UE), that is:*

1. $(\alpha\|\beta) \wedge \gamma \dashv\vdash_{\mathsf{DAL}} (\alpha \wedge \gamma)\|(\beta \wedge \gamma)$
2. $(\alpha\|\beta) \vee \gamma \dashv\vdash_{\mathsf{DAL}} (\alpha \vee \gamma)\|(\beta \vee \gamma)$
3. $\neg(\alpha\|\beta) \dashv\vdash_{\mathsf{DAL}} (\neg\alpha)\|(\neg\beta)$
4. $\alpha \wedge \beta \vdash_{\mathsf{DAL}} \alpha\|\beta \vdash_{\mathsf{DAL}} \alpha \vee \beta$
5. $\alpha\|\beta \vdash_{\mathsf{DAL}} (\alpha \vee \gamma)\|(\beta \vee \delta)$

These are all straightforward to verify with Lemma 12, same for ($\|$assoc). Now these results together with the Fundamental Theorem already entail the following:

Lemma 15 *The logic* DAL *is not closed under uniform substitution.*

For example, $p \vee \neg p$ is a theorem in DAL. However, if we uniformly substitute p by $p\|q$, we obtain $(p\|q) \vee \neg(p\|q)$. We have

(12) $\blacksquare((p\|q) \vee \neg(p\|q)) = (\blacksquare p\|q) \vee (\neg\blacklozenge p\|q) = (p \wedge q) \vee \neg(p \vee q)$

which is not a classical theorem, hence $\not\vdash_{\mathsf{DAL}} (p\|q) \vee \neg(p\|q)$. It is easy to see the trick: negation changes $\blacksquare$ to $\blacklozenge$.

DAL as a logic of distrust Now we can explain why this logic models distrustful reasoning. Basically, in DAL, there is no *generally valid shape* of an argument. In order to verify the validity of an argument, we first have to look inside all of its constituents. Modus ponens, widely acknowledged as the most basic inference rule, is not generally valid in DAL:

$$\not\Vdash_{\mathsf{DAL}} (p\|q) \to r,\ p\|q \vdash r$$

It is an easy exercise to verify this; the reason is the same as below Lemma 15: define, as usual, $(p\|q) \to r \equiv \neg(p\|q) \vee r$, hence

(13) $\blacklozenge((p\|q) \to r,\ p\|q) = (p \wedge q) \to r,\ p \vee q$

which obviously does not imply r. Conceptually, this reflects distrust: maybe the formula $p\|q$ is *intended in a different sense* than the premise of $(p\|q) \to r$, hence MP is not applicable. In other words, we do not trust that ambiguous terms are used consistently in an argument. Consider the following argument:

(14) `Peter loves plants. If someone loves plants, he loves nature.`
$\therefore$ `Peter loves nature`

This is not necessarily correct, since `plant` is famously ambiguous. But if someone makes this argument, it takes a portion of mistrust to refuse it, since you are more or less implying that the person might be willingly misleading you. Put simply: DAL is a good logic to reason with the devil,[10] but maybe not a good one to reason with your friends and family. It seems problematic in linguistic applications, as lexical word meanings are always somewhat opaque, and ambiguity can never be completely excluded (just think of metaphors). On the other side, if arguments/statements are patched together from different and independent sources, DAL might be appropriate, because even if there is no ill will, how should one guarantee uniform usage?[11]

Lemma 12 also shows another property of DAL: for every formula α, we have

(15) $\alpha \dashv\vdash_{\mathsf{DAL}} \blacklozenge\alpha\|\blacksquare\alpha$

This follows in a straightforward fashion form Lemma 12. This means: every formula in DAL is equivalent to a *binary* ambiguity. Even for a logic as simple

[10] People often include here lawyers and politicians, but I guess that depends.
[11] This might be the case for entries in ontologies, see [1].

and intuitive as DAL, there are some counterintuitive results. The most striking one is that all formulas of the form $a\|\neg a$, a an arbitrary classical formula, are equivalent. In particular, this means we derive sequents like $p\|\neg p \vdash_{\mathsf{DAL}} q\|\neg q$. Since equivalence entails congruence for DAL, we can also derive sequents like

(16) $p\|\neg p\|\neg p \dashv\vdash_{\mathsf{DAL}} q\|\neg q\|\neg p$

This can be generalized to the following observation: assume β is unambiguous, and $\blacksquare\alpha \vdash_{\mathsf{CL}} \beta \vdash_{\mathsf{CL}} \blacklozenge\alpha$. Then $\alpha \equiv_{\mathsf{DAL}} \alpha\|\beta$. This has another important consequence:

Lemma 16 *1.* DAL *does not satisfy the general law of disambiguation.*

2. DAL *satisfies the weak law of disambiguation,* $\alpha\|b\|\gamma, \neg b \vdash \alpha\|\gamma$*, where* b *is a formula of classical logic.*

Proof. 1. Take the sequent $(p\|\neg p\|q) \wedge \neg(p\|\neg p) \vdash q$, which is an instance of the law of disambiguation. This is *not* valid in DAL (as can be easily checked).

2. DAL satisfies ($\|$mon), and if b is classical, $b \wedge \neg b$ is a contradiction, so

(17) $(\alpha\|b\|\gamma) \wedge \neg b \equiv_{\mathsf{DAL}} (\alpha \wedge \neg b)\|(b \wedge \neg b)\|(\gamma \wedge \neg b) \vdash_{\mathsf{DAL}} \alpha\|\alpha\|\gamma \equiv_{\mathsf{DAL}} \alpha\|\gamma$

$\dashv$

Lemma 16.1 is slightly surprising. It is an important result because it proves that the law of disambiguation is independent from other properties of ambiguity. We see how the law of disambiguation in a sequent like (17) depends on the fact whether 1. $b \wedge \neg b$ is a contradiction, and 2. whether ($\|$mon) is holds. Since in DAL $\beta \wedge \neg\beta$ is not generally a a contradiction, the general LoD does not hold. But $\beta \wedge \neg\beta$ is a contradiction in all trustful ambiguity logics, hence they usually satisfy the general (LoD). We will later see the logic $L_{\blacksquare\blacklozenge}$, which does not satisfy ($\|$mon) and in fact does not even satisfy (wLoD). This should also illustrate how the general LoD cannot be a necessary criterion to define a "logic of ambiguity": it would in fact exclude most distrustful logics!

We present one more result that we will need later on. The operator $\blacklozenge$ obviously makes every proposition "truer", the operator $\blacksquare$ makes it "less true". Hence if $\blacklozenge\alpha$ is false in a model, then $\blacksquare\alpha$ is also false, and if $\blacksquare\alpha$ is true in a model, then $\blacklozenge\alpha$ is also true. What is less obvious is that $\blacklozenge$ preserves theoremhood, and dually $\blacksquare$ preserves contradictions, even when we close under u-substitution.

Lemma 17 *Assume* $\alpha \in Form(\mathsf{CL})$*,* $\sigma : Var \to Form(\mathsf{AL})$ *is a uniform substitution. Then there is a function* $f : \wp(Var) \to \wp(Var)$ *such that*

1. *if* $M \not\models \blacklozenge\sigma(\alpha)$*, then* $f(M) \not\models \alpha$
2. *if* $M \models \blacksquare\sigma(\alpha)$*, then* $f(M) \models \alpha$

Proof. Define $f(M) := \{p : M \models \blacklozenge\sigma(p)\}$. We prove that this does the job, via simultaneous induction (for both 1. and 2.) on formula complexity $|\alpha|$.

Base case: $\alpha = p$.

1. Contraposition: assume $f(M) \models p$. Then by definition $M \models \blacklozenge\sigma(p)$.

2. Assume $M \models \blacksquare\sigma(p)$. Hence $M \models \blacklozenge\sigma(p)$. Hence $p \in f(M)$, so $f(M) \models p$.

Induction step. Assume the claims 1.,2. hold for α, β.

$\alpha \wedge \beta$.

1. Assume there is an M such that $M \not\models \blacklozenge\sigma(\alpha \wedge \beta)$ $(= (\blacklozenge\sigma\alpha) \wedge (\blacklozenge\sigma\beta))$. Case i: $M \not\models \blacklozenge\sigma(\alpha)$. Then the claim follows by IH. Case ii: parallel.

2. Assume that $M \models \blacksquare\sigma(\alpha \wedge \beta)$ $(= \blacksquare\sigma(\alpha) \wedge \blacksquare\sigma(\beta))$. Hence $f(M) \models \alpha$, $f(M) \models \beta$, hence $f(M) \models \alpha \wedge \beta$.

$\alpha \vee \beta$ Dual.

$\neg\alpha$

1. Assume that $M \not\models \blacklozenge\sigma(\neg\alpha)$ $(= \neg\blacksquare\sigma(\alpha))$. Hence $M \not\models \neg\blacksquare\sigma(\alpha)$, so $M \models \blacksquare\sigma(\alpha)$. By IH2, $f(M) \models \alpha$, hence $f(M) \not\models \neg\alpha$.

2. Dual. ⊣

Corollary 18 *If $\alpha \in Form(\mathsf{CL})$, $\sigma : Var \to Form(\mathsf{AL})$ is a uniform substitution, then*

1. *if α is a theorem, then $\blacklozenge\sigma(\alpha)$ is a theorem in* DAL.
2. *if α is a contradiction, then $\blacksquare\sigma(\alpha)$ is a contradiction in* DAL.

Proof of this is a simple contraposition using the previous lemma. So $\blacklozenge\sigma(\alpha)$ does not preserve truth of α in a model (provided α is not a theorem), but it preserves theoremhood.

To sum up, DAL is a very simple and natural logic of ambiguity, and many results can be obtained as easy exercise. This does not make it less interesting though, and we think that it might be useful for some applications. Another big advantage is that it is straightforward to extend it to predicate logic (or fragments thereof, like description logic), which seems a prerequisite for real-world applications. DAL seems to be central also for another reason: we will see that it "pops up" as the inner logic both of cTAL and $L_{\blacklozenge\blacksquare}$, the minimal ambiguity logic!

3.5 Trust with Order: cTAL and $L_{\blacksquare\blacklozenge}$

We have stated here (and argued extensively in [18]) that TAL is a very reasonable ambiguity logic of trust. We will now show the same is *not* true for cTAL: cTAL is not a reasonable ambiguity logic, since it is way too permissive. This makes a more general point: trustful reasoning does not go well together with $\|$-commutativity. If we trust in uniform usage, we should keep track of ordering (of plausibility). The argument is purely formal though, rather than conceptual. Consider the following rules.

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta \quad \Theta_1, \Theta_2 \vdash \Delta}{(\Gamma_1; \Theta_1), (\Gamma_2; \Theta_2) \vdash \Delta}\ (;\, I) \qquad \frac{\Delta \vdash \Gamma_1, \Gamma_2 \quad \Delta \vdash \Theta_1, \Theta_2}{\Delta \vdash (\Gamma_1; \Theta_1), (\Gamma_2; \Theta_2)}\ (I;\,)$$

In [18] these were part of the calculus; however, we can show the following:

Lemma 19 *In* TAL, cTAL*, the rules* $(;\, I), (I;\,)$ *are admissible.*

Proof. In TAL and cTAL, $\Gamma \vdash \Delta$ iff $l(\Gamma) \vdash r(\Delta)$. Hence $\Gamma_1, \Gamma_2 \vdash \Delta$ entails $\Gamma_1, l(\Gamma_2) \vdash \Delta$ entails $\Gamma_1 \vdash \Delta, \neg(l(\Gamma_2))$, and $\Theta_1, \Theta_2 \vdash \Delta$ entails $\Theta_1, l(\Theta_2) \vdash \Delta$ entails $\Theta_1 \vdash \Delta, \neg(l(\Theta_2))$

We derive $\Gamma_1; \Theta_1 \vdash \Delta, (\neg l(\Gamma_2); \neg l(\Theta_2))$ and $(\Gamma_1; \Theta_1), (\neg\neg l(\Gamma_2); \neg\neg l(\Theta_2)) \vdash \Delta$, and by double negation congruence (see Lemma 32), the claim follows. ⊣

In cTAL we can then make the following proof:

$$\dfrac{\dfrac{\dfrac{\alpha, \beta \vdash \alpha \wedge \beta \quad \beta, \alpha \vdash \alpha \wedge \beta}{(\alpha; \beta), (\beta; \alpha) \vdash \alpha \wedge \beta}\,(\mathrm{I};\,)}{(\alpha; \beta), (\alpha; \beta) \vdash \alpha \wedge \beta}\,(;\ \mathrm{comm})}{\alpha; \beta \vdash \alpha \wedge \beta}\,(;\ \mathrm{contr})$$

The same works on the right, obviously. In this sense, we can say that cTAL is **too trustful**: these sequents are clearly counterintuitive. Interestingly, even though we can derive $a \vee b \vdash_{\mathsf{cTAL}} a \| b$ and $a \| b \vdash_{\mathsf{cTAL}} a \wedge b$, the logic is still consistent and conservatively extends classical logic, because of the absence of (cut); we cannot derive $a \vee b \vdash a \wedge b$. To see where the logic "goes wrong", consider that in TAL, we can derive the weaker

$$(a \| b) \wedge (b \| a) \vdash_{\mathsf{TAL}} a \wedge b$$

This does not strike us as necessarily incorrect: basically if the first reading can be a and b, and the second reading can be a and b, $a \wedge b$ follows. This becomes clearer if we "distribute out" the sequent.

(18) $(a \| b) \wedge (b \| a) \equiv_{\mathsf{TAL}} (a \wedge (b \| a)) \| (b \wedge (b \| a)) \equiv_{\mathsf{TAL}} (a \wedge b) \| (a \wedge a) \| (b \wedge b) \| (a \wedge b)$

Here we see the same pattern as in the tautology $(a \| b) \vee \neg(a \| b)$ "distributed out" in (8), where only the margins are tautologies. This is acceptable, but together with (; comm),(, contr), we obtain strange consequences. Now it comes even worse for cTAL. Consider the following lemma, which holds only for cTAL, not for TAL. It is subtle, but it will make a big difference:

Lemma 20 *In* cTAL, $\Gamma[\alpha]; \Gamma[\beta] \vdash \Delta$ *iff* $\Gamma[\alpha; \beta] \vdash \Delta$*. Dually on the right.*

Proof. By induction on complexity of $\Gamma[-]$. Base case (identity function) is clear, since then $\alpha; \beta = \Gamma[\alpha]; \Gamma[\beta] = \Gamma[\alpha; \beta]$. Now assume it holds for some $\Gamma[-]$.

, $(\Gamma[\alpha], \Gamma'); (\Gamma[\beta], \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ iff $(\Gamma[\alpha]; \Gamma[\beta]), \Gamma', \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by 2× (distr/invDistr)) iff $(\Gamma[\alpha]; \Gamma[\beta]), \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by (, contr)) iff $\Gamma[\alpha; \beta], \Gamma' \vdash_{\mathsf{cTAL}} \Delta$ (by IH). Same on the right.

; $(\Gamma[\alpha]; \Gamma'); (\Gamma[\beta]; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ iff $(\Gamma[\alpha]; \Gamma[\beta]; \Gamma'; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ (by (; comm) iff $(\Gamma[\alpha; \beta]; \Gamma') \vdash_{\mathsf{cTAL}} \Delta$ (by (; contr)). Same on the right. ⊣

Now consider the following rules of ambiguous weakening:

$$\text{(; weak)}\ \frac{\Gamma[\Theta] \vdash \Delta}{\Gamma[(\Theta;\ \Psi)] \vdash \Delta} \qquad \text{(weak;)}\ \frac{\Gamma \vdash \Delta[\Theta]}{\Gamma \vdash \Delta[(\Theta;\ \Psi)]}$$

Lemma 21 *(; weak), (weak;) are admissible in* cTAL.

Proof. Assume we have $\Gamma[\Theta] \vdash_{\mathsf{cTAL}} \Delta$. We can apply (, weak) and obtain:

$$\dfrac{\dfrac{\dfrac{\Gamma[\Psi],\ \Gamma[\Theta] \vdash \Delta \quad \Gamma[\Theta],\ \Gamma[\Psi] \vdash \Delta}{(\Gamma[\Psi];\ \Gamma[\Theta]),\ (\Gamma[\Theta];\ \Gamma[\Psi]) \vdash \Delta}\ (;\ \mathrm{I})}{(\Gamma[\Psi];\ \Gamma[\Theta]),\ (\Gamma[\Psi];\ \Gamma[\Theta]) \vdash \Delta}\ (;\ \mathrm{comm})}{(\Gamma[\Psi];\ \Gamma[\Theta]) \vdash \Delta}\ (,\ \mathrm{contr})$$

Finally, we obtain $\Gamma[\Psi;\ \Theta] \vdash_{\mathsf{cTAL}} \Delta$ by the previous lemma. ⊣

Now this is of course *way* too trustful: it entails that $a_1\|...\|a_i \vdash_{\mathsf{cTAL}} b_1\|...\|b_j$ if there is at least one pair (n, m) such that $a_n \vdash_{\mathsf{CL}} b_m$ (this is a sufficient criterion, not necessary). Recall the definition of ambiguous normal form $\mathit{anf}(\alpha)$ (Definition 1).

Lemma 22 *For all* $\alpha \in \mathit{Form}(\mathsf{AL})$ *and* $\gamma \in \mathit{anf}(\alpha)$, $\blacksquare\alpha \equiv_{\mathsf{CL}} \blacksquare\gamma$, $\blacklozenge\alpha \equiv_{\mathsf{CL}} \blacklozenge\gamma$.

Proof is an easy exercise using DeMorgan laws and $\wedge, \vee$-idempotence. From here, we obtain the following:

Lemma 23 $\blacksquare\alpha \leqq_{\mathsf{TAL}} \alpha \leqq_{\mathsf{TAL}} \blacklozenge\alpha$. *Same for* cTAL

This follows from the fact that 1. $\alpha \equiv_{\mathsf{TAL}} a_1\|...\|a_i$ (universal distribution, Lemma 32), 2. $a_1 \wedge ... \wedge a_i \equiv_{\mathsf{TAL}} \blacksquare\alpha$ (Lemma 22), and 3. $a_1 \wedge ... \wedge a_i \leqq_{\mathsf{TAL}} a_1\|...\|a_i$ (see Lemma 32). Same for $\blacklozenge$.

Lemma 24 $\alpha \vdash_{\mathsf{cTAL}} \beta$ *iff* $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$

Proof. *Only if* Follows from the fact that $\blacksquare\alpha \leqq_{\mathsf{cTAL}} \alpha$, $\beta \leqq_{\mathsf{cTAL}} \blacklozenge\beta$.

If: Assume $\alpha = a_1\|...\|a_i$, $\beta = b_1\|...\|b_j$ are in ambiguous normal form (all a, b classical). This comes without loss of generality by from Lemma 22 and Lemma 32.

$\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ means then as much as $a_1,\ ...,\ a_i \vdash_{\mathsf{CL}} b_1,\ ...,\ b_j$. By conservative extension, we have $a_1,\ ...,\ a_i \vdash_{\mathsf{cTAL}} b_1,\ ...,\ b_j$.

Now we can apply (; weak) to every formula of the sequent, until we obtain $\underbrace{\alpha,\ ...,\ \alpha}_{i \text{ times}} \vdash_{\mathsf{cTAL}} \underbrace{\beta,\ ...,\ \beta}_{j \text{ times}}$; then we can apply (, contr) to obtain $\alpha \vdash_{\mathsf{cTAL}} \beta$. ⊣

This means: our logic cTAL has a simple characterization in terms of classical logic! The same does not apply, however, to TAL, at least as far as we can see. This already entails the following:

- In cTAL, (classic cut) (unambiguous formula in unambiguous context) is admissible: if α is classical, $\Gamma \vdash_{\mathsf{cTAL}} \alpha$ and $\Delta, \alpha \vdash_{\mathsf{cTAL}} \Theta$ entail $\Gamma, \Delta \vdash_{\mathsf{cTAL}} \Theta$. (Same holds for TAL, see [18].)

- but cut in general or even transitivity of inference are not sound: $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ and $\blacksquare\beta \vdash_{\mathsf{CL}} \blacklozenge\gamma$ does *not* entail that $\Vdash_{\mathsf{CL}} \blacksquare\alpha \vdash \blacklozenge\gamma$

Note that (classic cut) and the fact that $p \vee \neg p \vdash_{\mathsf{cTAL}} p\|\neg p$ entail that $p\|\neg p$ is a *theorem* of cTAL, and, by a dual argument, at the same time a *contradiction*! The following lemma is now obvious:

Corollary 25 $\mathsf{DAL} \subseteq \mathsf{cTAL}$

Proof. $\blacksquare\beta \vdash_{\mathsf{CL}} \blacklozenge\beta$, hence if $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, then $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$, so $\alpha \vdash_{\mathsf{cTAL}} \beta$. ⊣

So in this case, trust does not only come with closure under substitution, but with more effectively valid inferences. We will generalize this observation in the Trust Theorem. Note also the following corollary:

Corollary 26 cTAL *is trivial.*

We have $\blacksquare(\alpha\|\beta) \vdash_{\mathsf{CL}} \blacklozenge(\alpha\|\gamma)$ for arbitrary α, β, γ, so the claim is obvious. To sum up this discussion, we can say the following: trust goes well together with an ordering of ambiguities (which corresponds to the likelihood ordering of options). Maybe this is even the *essence of trust* (in ambiguity): that we have to assume a fixed ordering of plausibility of reading; since otherwise, trust becomes ingenuousness.

Finally, let us make the following observation: both DAL and cTAL have simple truth-theoretic characterizations, whereas their non-commutative counterparts $\bar{\mathsf{c}}\mathsf{DAL}$, TAL require a complex proof-theory. However, their proof-theories seem to have some "natural" properties, if adding commutativity makes them become so easily accessible in terms of truth (this relates to a property which we will call *classically congruent* later, see Definition 46). We think that this, among other, indicates their particular importance and "naturalness".

4 Congruence Algebras and Inner Logics

4.1 Congruence Algebras and Generalized Boolean Algebras

The Fundamental Theorem stated that we cannot think algebraically about reasoning with ambiguity, since our logics, in order to be non-trivial, have to abandon either closure under substitution of equivalents or closure under uniform substitution, both intrinsic features of algebra. This however does *not* prevent us from analyzing ambiguity logics algebraically. In particular, we can construct, for every ambiguity logic L, a **congruence algebra**, which is similar to the well-known Lindenbaum-Tarski algebra, but its elements are *not* $\dashv\vdash_L$-equivalence classes, but rather $\equiv_L$**-congruence classes** (the relations coincide for most logics of distrust, though). Recall that for every logic L, the outer logic $\vdash_L$ gives rise to the inner logic $\leqq_L$, where $\alpha \leqq_L \beta$ means α is *weaker in all logical contexts* than β, and to its reflexive closure $\equiv_L$, which means equivalence in all logical contexts.

For every ambiguity logic L, $\equiv_L$ is obviously an equivalence relation, hence we can define the algebra of its congruence classes, or its **congruence algebra**:

$$Cong(L) := (Form(\mathsf{AL})_{\equiv_L}, \wedge, \vee, \neg, \|)$$

We know that logical connectives are well-defined operators on congruence classes, independently of representatives (whereas this is not satisfied for $\dashv\vdash_L$!). The Fundamental Theorem 2 in this sense obtains a different reading/meaning: for all non-trivial ambiguity logics, their congruence algebra will *not be Boolean.*

Corollary 27 *Let L be a logic of ambiguity. Assume the Boolean reduct of $Cong(L)$ is a Boolean algebra. Then L is trivial.*

This is proved in [18], Theorem 75. Obviously, $Cong(L)$ can be equipped with an order relation $\leq_{Cong(L)}$, where $\alpha \leq_{Cong(L)} \beta$ iff $\alpha \wedge \beta \equiv_L \alpha$ and $\alpha \vee \beta \equiv_L \beta$. The following notion is an important criterion for "plausibility" of logics, which basically states that $\wedge, \vee$ behave in a "logical way":

Definition 28 *A logic L is **naturally ordered** if $\alpha \leq_{Cong(L)} \beta$ iff $\alpha \leqq_L \beta$.*

This notion is very important in our view. Note that this is not equivalent with requiring that $Cong(L)$ is lattice ordered: $Cong(L)$ can be lattice ordered, where $\alpha \leq_{Cong(L)} \beta$. Hence $\Gamma[\beta] \vdash \Delta$ entails $\Gamma[\alpha \vee \beta] \vdash \Delta$. This need not entail $\Gamma[\alpha] \vdash \Delta$, because ($\vee$I) need not be invertible (for example, TAL without (inter2)). Conversely, a logic can be naturally ordered, yet we might have $\alpha \wedge \alpha \lneqq_{Cong(L)} \alpha$ ($\wedge, \vee$ need not be idempotent). Every logic we have considered until now is both lattice ordered and naturally ordered, but we will see counterexamples below. For now we focus on logics satisfying these properties.

The algebraic study of naturally lattice ordered ambiguity logics is basically (an application of) the study of generalized Boolean algebras, a well-established algebraic field which explores the space between distributive lattices with a unary operator and Boolean algebras. So let us have a look at some classes of generalized Boolean algebras.

Definition 29 *A **generalized Boolean algebra** (GBA) $(B, \wedge, \vee, \sim, 0, 1)$ is an algebra where $(B, \wedge, \vee, 0, 1)$ is a bounded, distributive lattice, and $\sim$ a unary function.*

Since every GBA is lattice ordered, the ordering $\leq$ is defined implicitly by $\wedge$ and $\vee$. $(B, \wedge, \vee, \sim, 0, 1)$ is

1. an **Ockham algebra**, if it satisfies $\sim(a \wedge b) = \sim a \vee \sim b$, $\sim(a \vee b) = \sim a \wedge \sim b$, and $\sim 0 = 1$, $\sim 1 = 0$.
2. a **DeMorgan algebra**, if in addition it satisfies $\sim\sim x = x$.
3. a **Kleene algebra**, if in addition it satisfies $x \wedge \sim x \leq y \vee \sim y$.

We say $\sim$ is a pseudocomplement if $\sim x = max\{x : x \wedge \sim x = 0\}$. Every pseudocomplemented de Morgan algebra is a Boolean algebra.[12]

[12]There exists the dual pseudocomplement, and dually each dual pseudocomplemented de Morgan algebra is a Boolean algebra.

4.2 On the Congruence Algebra of TAL

We will see that the congruence algebra of TAL is a Kleene algebra. At the same time, for all known classes of generalized Boolean algebras, Kleene algebras seem to be the most specific class in which the congruence algebra of TAL is contained. We now prove some central congruences. To this aim, we first show the following Context Lemma, which allows us to easily establish congruence for arbitrary contexts.

We say that a context function $\Gamma[-]$ is atomic, if $\Gamma[\alpha] = (\Gamma_1, \alpha, \Gamma_2)$, or $\Gamma[\alpha] = (\Gamma_1; \alpha; \Gamma_2)$. In the first case, we say the atomic function is classical, in the latter we say it is ambiguous. Note that the identity function is atomic and both classical and ambiguous. Obviously, every context function $\Psi[-]$ can be brought into the form

(19) $\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-] \circ \Delta_i[-] \circ \Gamma_{i+1}[-]$

where each $\Gamma_j[-]$ is a classic context function, each $\Delta_j[-]$ an ambiguous context function, and $\circ$ denotes function composition.

Lemma 30 *(Context Lemma)*

1. *Assume for all contexts* $\Gamma, \Delta,\ \Gamma, \beta \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$. *Then for all context-functions* $\Gamma[-]$, *contexts* $\Delta,\ \Gamma[\beta] \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$
2. *Assume for all contexts* $\Gamma, \Delta,\ \Delta \vdash_{\mathsf{TAL}} \Gamma, \beta$ *entails* $\Delta \vdash_{\mathsf{TAL}} \Gamma, \alpha$. *Then for all contexts* Γ, *context-functions* $\Delta[-],\ \Gamma \vdash_{\mathsf{TAL}} \Delta[\beta]$ *entails* $\Gamma \vdash_{\mathsf{TAL}} \Delta[\alpha]$

We only prove 1., since 2. is completely dual.

Proof. We make an induction over the complexity of context functions, that is i in (19).

Base case is $i = 0$, which is trivial (identity function). Assume (IH) the claim holds for some i, and we have

$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-] \circ \Delta_{i+1}[-] \circ \Gamma_{i+1}[-](\beta)$	$\vdash_{\mathsf{TAL}} \Psi$.	Then
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, B); C)$	$\vdash_{\mathsf{TAL}} \Psi$	(alt. notation)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, \alpha, B); C)$	$\vdash_{\mathsf{TAL}} \Psi$	(, weak)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](\beta, (A; (\alpha, B); C))$	$\vdash_{\mathsf{TAL}} \Psi$	(invDistr)
$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](\alpha, (A; (\alpha, B); C))$	$\vdash_{\mathsf{TAL}} \Psi$	(by IH)

Now we can apply (inter2) to obtain the desired result:

$$\frac{\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](\alpha, (A; (\alpha, B); C)) \vdash \Psi \quad \Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-](A; (\beta, \alpha, B); C)] \vdash \Psi}{\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_i[-]((A; (\alpha, B); C)) \vdash \Psi}$$

which is an alternative notation for

$$\Gamma_1[-] \circ \Delta_1[-] \circ ... \circ \Gamma_{i-1}[-] \circ \Delta_{i+1}[-] \circ \Gamma_{i+1}[-](\alpha) \vdash \Psi \qquad \dashv$$

This lemma is very useful: many congruence proofs, which can be tedious inductions over contexts or proof lengths, become simple exercises. For example, from our general invertibility lemma, we know that $\Gamma, \neg\alpha \vdash_{\mathsf{TAL}} \Delta$ if and only if $\Gamma \vdash_{\mathsf{TAL}} \alpha, \Delta$. By iterating this two times, we obtain the first result; also the following are simple exercises to prove.

Lemma 31 *1.* $\Gamma, \neg\neg\alpha \vdash_{\mathsf{TAL}} \Delta$ *iff* $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$

2. $\Gamma, (\alpha\|\beta) \wedge \gamma \vdash_{\mathsf{TAL}} \Delta$ *iff* $\Gamma, (\alpha \wedge \gamma)\|(\beta \wedge \gamma) \vdash_{\mathsf{TAL}} \Delta$

3. $\Gamma, (\alpha\|\beta) \vee \gamma \vdash_{\mathsf{TAL}} \Delta$ *iff* $\Gamma, (\alpha \vee \gamma)\|(\beta \wedge \gamma) \vdash_{\mathsf{TAL}} \Delta$

4. $\Gamma, \alpha \vee \beta \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma, \alpha\|\beta \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma, \alpha \wedge \beta \vdash_{\mathsf{TAL}} \Delta$

This and many similar results (which are easy exercises) entail by means of Lemma 30 the following (mostly also proved in [18]).

Lemma 32 *In* TAL*, for all* α, β, γ*, we have:*

1.	$(\alpha\|\beta) \wedge \gamma \equiv_{\mathsf{TAL}} (\alpha \wedge \gamma)\|(\beta \wedge \gamma)$	*5.*	$\alpha\|\beta \leqq_{\mathsf{TAL}} (\alpha \vee \gamma)\|(\beta \vee \delta)$
2.	$(\alpha\|\beta) \vee \gamma \equiv_{\mathsf{TAL}} (\alpha \vee \gamma)\|(\beta \vee \gamma)$	*6.*	$\alpha \equiv_{\mathsf{TAL}} \neg\neg\alpha$
3.	$\neg(\alpha\|\beta) \equiv_{\mathsf{TAL}} (\neg\alpha)\|(\neg\beta)$	*7.*	$(\alpha \vee \beta) \wedge \gamma \equiv_{\mathsf{TAL}} (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$
4.	$\alpha \wedge \beta \leqq_{\mathsf{TAL}} \alpha\|\beta \leqq_{\mathsf{TAL}} \alpha \vee \beta$	*8.*	$\neg(\alpha \wedge \beta) \equiv_{\mathsf{TAL}} \neg\alpha \vee \neg\beta$

Lemma 33 *If* $\alpha \in anf(\beta)$*, we have* $\alpha \equiv_{\mathsf{TAL}} \beta$

Moreover, we obtain a very nice result, which will have some importance later on. Recall that (classic cut), that is (cut) for classical cut formulas in unambiguous context, is admissible in TAL (for explicit proof, see [18]).

Lemma 34 *Assume* $\alpha \vdash_{\mathsf{CL}} \beta$*. Then* $\alpha \leqq_{\mathsf{TAL}} \beta$*.*

Proof. From $\Gamma, \beta \vdash_{\mathsf{TAL}} \Delta$ and $\alpha \vdash_{\mathsf{CL}} \beta$ we can deduce $\Gamma, \alpha \vdash_{\mathsf{TAL}} \Delta$ by (classic cut). Same goes for $\Gamma \vdash_{\mathsf{TAL}} \Delta, \alpha$ and $\alpha \vdash_{\mathsf{CL}} \beta$, which entail $\Gamma \vdash_{\mathsf{TAL}} \Delta, \beta$. Hence we simply apply the Context Lemma, and $\alpha \leqq_{\mathsf{TAL}} \beta$ follows. ⊣

Recall the maps l, r from contexts to formulas. We define complexity of contexts as follows: for all formulas α, $|\alpha| = 1$; $|\Gamma, \Delta| = |\Gamma| + |\Delta|$ and $|\Gamma; \Delta| = |\Gamma| + |\Delta|$. We write $\Gamma \equiv_l \Delta$ if $\Theta, \Gamma \vdash \Psi$ iff $\Theta, \Delta \vdash \Psi$; parallel for $\equiv_r$.

Lemma 35 *For all contexts* Ξ*,*

1. $\neg\Xi \equiv_r \neg l(\Xi)$

2. $\neg\Xi \equiv_l \neg r(\Xi)$

Proof. We only prove claim 1. To increase readability, we put $a := l(A)$, $b := l(B)$.

Induction over $|\Xi|$. If $\Xi = \xi$ is a formula, the claim is obvious. Assume it holds for all $\Xi : |\Xi| \leq n$ for a given n. We distinguish two cases

Case 1. $\Xi = (A, B)$. Note that $\neg l(A, B) = \neg(l(A) \wedge l(B)) =: \neg(a \wedge b)$.

Assume	$\Gamma \vdash \neg(a \wedge b), \Delta$	Assume	$\Gamma \vdash \Delta, \neg A, \neg B$
Hence	$\Gamma, (a \wedge b) \vdash \Delta$	Hence	$\Gamma \vdash \Delta, \neg a, \neg b$
Hence	$\Gamma, a, b \vdash \Delta$	Hence	$\Gamma, a, b \vdash \Delta$
Hence	$\Gamma \vdash \Delta, \neg a, \neg b$	Hence	$\Gamma, (a \wedge b) \vdash \Delta$
Hence	$\Gamma \vdash \Delta, \neg A, \neg B$	Hence	$\Gamma \vdash \neg(a \wedge b), \Delta$
Hence	$\Gamma \vdash \Delta, \neg(A, B)$	Hence	$\Gamma \vdash \neg l(A, B), \Delta$

Case 2. $\Xi = (A;\ B)$. Note that $\neg l(A;\ B) = \neg(l(A)\|l(B)) =: \neg(a\|b)$.

Assume	$\Gamma \vdash \neg(a\|b),\ \Delta$	Assume	$\Gamma \vdash \neg(A;\ B),\ \Delta$
Hence	$\Gamma,\ a\|b \vdash \Delta$	Hence	$\Gamma \vdash (\neg A;\ \neg B),\ \Delta$
Hence	$\Gamma,\ (a;\ b) \vdash \Delta$	Hence	$\Gamma \vdash (\neg a;\ \neg b),\ \Delta$
Hence	$\Gamma \vdash (\neg a;\ \neg b),\ \Delta$	Hence	$\Gamma,\ (\neg\neg a;\ \neg\neg b) \vdash \Delta$
Hence	$\Gamma \vdash (\neg A;\ \neg B),\ \Delta$	Hence	$\Gamma,\ (a;\ b) \vdash \Delta$
Hence	$\Gamma \vdash \neg(A;\ B),\ \Delta$	Hence	$\Gamma,\ (a\|b) \vdash \Delta$
		Hence	$\Gamma \vdash \neg(a\|b),\ \Delta$

⊣

This gives us the following nice corollary:

Lemma 36 *(Generalized negation)*

1. $\Gamma,\ \Xi \vdash_{\mathsf{TAL}} \Delta$ *entails* $\Gamma \vdash_{\mathsf{TAL}} \neg\Xi,\ \Delta$
2. $\Gamma \vdash_{\mathsf{TAL}} \Xi,\ \Delta$ *entails* $\Gamma,\ \neg\Xi \vdash_{\mathsf{TAL}} \Delta$

Proof. Assume $\Gamma,\ \Xi \vdash_{\mathsf{TAL}} \Delta$. Hence $\Gamma,\ l(\Xi) \vdash_{\mathsf{TAL}} \Delta$, so $\Gamma \vdash_{\mathsf{TAL}} \neg l(\Xi),\ \Delta$, hence $\Gamma \vdash_{\mathsf{TAL}} \neg\Xi,\ \Delta$. ⊣

We will negate not only contexts, but also context functions, as in $(\neg\Gamma)[-]$, with the obvious meaning. Note that $(\neg\Gamma)[\alpha] = (\neg\Gamma[-])(\alpha) \neq \neg(\Gamma[\alpha]) = (\neg\Gamma)[\neg\alpha]$. The law of contraposition holds in *Cong*(TAL):

Lemma 37 $\alpha \leqq \beta$ *iff* $\neg\beta \leqq \neg\alpha$

Proof. $\Rightarrow$ Assume $\alpha \leqq \beta$, and $\Gamma[\neg\alpha] \vdash_{\mathsf{TAL}} \Delta$. Then $\vdash_{\mathsf{TAL}} \neg(\Gamma[\neg\alpha]),\ \Delta$, hence by DN-congruence $\vdash_{\mathsf{TAL}} \neg(\Gamma)[\alpha],\ \Delta$. So by assumption $\vdash_{\mathsf{TAL}} (\neg\Gamma)[\beta],\ \Delta$, hence $(\neg\neg\Gamma)[\neg\beta] \vdash_{\mathsf{TAL}} \Delta$, so $\Gamma[\neg\beta] \vdash_{\mathsf{TAL}} \Delta$. Similar on the right hand side.

$\Leftarrow$ Immediate from double negation congruence. ⊣

Now we return to the question we have mentioned in the beginning: where can we locate *Cong*(TAL) in the family of generalized Boolean algebras? The following is already immediate:

Lemma 38 *The Boolean reduct of Cong*(TAL) *is a DeMorgan algebra.*

We now strengthen this result. Having double negation, DeMorgan law and the distributive laws for lattices, we are lacking the inequation

(20) $p \wedge {\sim}p \leq q \vee {\sim}q$

which in our terms means: every contradiction of TAL is logically stronger ($\leqq_{\mathsf{TAL}}$-smaller) than every theorem of TAL. Again, Lemma 30 allows us to prove this in a simple fashion:

Lemma 39 *1. Assume* $\Gamma[\Theta] \vdash_{\mathsf{TAL}} \Delta$ *and* $\Psi \vdash_{\mathsf{TAL}}$ *(*Ψ *is a contradiction). Then* $\Gamma[\Psi] \vdash_{\mathsf{TAL}} \Delta$ *is derivable.*

2. *Assume* $\Gamma \vdash_{\mathsf{TAL}} \Delta[\Theta]$ *and* $\vdash_{\mathsf{TAL}} \Psi$ *(Ψ is a theorem). Then* $\Gamma \vdash_{\mathsf{TAL}} \Delta[\Psi]$ *is derivable.*

Proof. We only prove 1. It is obvious that if $\Psi \vdash_{\mathsf{TAL}}$, we can derive $\Gamma, \Psi \vdash_{\mathsf{TAL}} \Delta$ (by weakening). So if $\Gamma, \Theta \vdash_{\mathsf{TAL}} \Delta$ and $\Psi \vdash_{\mathsf{TAL}}$, then $\Gamma, \Psi \vdash_{\mathsf{TAL}} \Delta$. From here, the claim follows via Lemma 30 (the Context Lemma). $\dashv$

From this double result, the following follows easily:

Corollary 40 *Let α be an arbitrary contradiction of* TAL*, β and arbitrary theorem. Then $\alpha \leqq_{\mathsf{TAL}} \beta$.*

Proof. $\Gamma[\beta] \vdash \Delta$ entails $\Gamma[\alpha] \vdash \Delta$ since α is a contradiction; $\Delta \vdash \Gamma[\alpha]$ entails $\Delta \vdash \Gamma[\beta]$ since β is a theorem. $\dashv$

Seemingly, the previous lemma even proves a stronger claim: theorems are maximal in the sense that one can substitute a theorem for any other formula on the right, and a contradiction for any other formula on the left. But we have to be careful: it does *not* mean this: not all theorems/contradictions are $\equiv_{\mathsf{TAL}}$-equivalent! Actually, this would entail that the TAL-congruence algebra is Boolean (follows from results on generalized Boolean algebras), and hence in an additional step, that TAL is trivial. Where is the mistake in that line of thought? All theorems are interchangeable on the right of $\vdash_{\mathsf{TAL}}$, all contradictions are interchangeable on the left. But: different theorems might have different distributions on the *left* of $\vdash_{\mathsf{TAL}}$, and contradictions on the *right*. In fact, Corollary 40 does not even entail that for some theorem β and an arbitrary α which is *not* a theorem, we have $\alpha \leqq_{\mathsf{TAL}} \beta$; in Corollary 40 we need the additional premise that α is a contradiction. We illustrate this with an example. Let T_1, T_2, T_3 be arbitrary theorems of TAL, $\bot$ an arbitrary contradiction. Obviously,

(21) $(T_1\| \bot) \vee \neg(T_1\| \bot) \equiv_{\mathsf{TAL}} (T_1 \vee \neg T_1)\|(\bot \vee\neg T_1)\|(T_1 \vee \neg \bot)\|(\bot \vee\neg \bot)$

are both theorems of TAL. Moreover, we have $(\bot \vee\neg T_1) \vdash_{\mathsf{TAL}}$. By two applications of (I; I) we can derive (abbreviated proof)

(22) $$\frac{(T_1 \vee \neg T_1) \vdash T_2 \quad (\bot \vee\neg T_1) \vdash \alpha \quad (T_1 \vee \neg \bot)\|(\bot \vee\neg \bot) \vdash T_3}{(T_1 \vee \neg T_1)\|(\bot \vee\neg T_1)\|((T_1 \vee \neg \bot)\|(\bot \vee\neg \bot)) \vdash_{\mathsf{TAL}} T_2\|\alpha\|T_3}$$

(for arbitrary α). However, we *cannot* derive (for arbitrary α)

(23) $p \vdash T_2\|\alpha\|T_3$, where p does not occur in $T_2\|\alpha\|T_3$

Assume we can. Then we can substitute $p \mapsto p\vee\neg p$, apply (classic cut) to obtain $\vdash T_2\|\alpha\|T_3$. Hence for all α, β, we would obtain $\beta \vdash T_2\|\alpha\|T_3$ (weakening), which would result in m-triviality (use (I; I) and (inter2)). Hence

(24) $p \not\leqq_{\mathsf{TAL}} (T_1\| \bot) \vee \neg(T_1\| \bot)$

whereas $p \leqq_{\mathsf{TAL}} p \vee \neg p$. So there are distinct congruence classes of theorems. We can however prove the following:

Lemma 41 *All classical theorems are* $\equiv_{\mathsf{TAL}}$ *equivalent and* $\leqq_{\mathsf{TAL}}$ *maximal. Dually for classical contradictions.*

Proof. We have already proved that every formula can be substituted by a theorem on the right. Let $T \in Form(\mathsf{CL})$ be a classical theorem and assume $\Gamma[T] \vdash_{\mathsf{TAL}} \Delta$. We obtain $\Gamma[T\mathbf{,}\ \alpha] \vdash_{\mathsf{TAL}} \Delta$ ($\mathbf{,}$ weak), $\Gamma[\alpha]\mathbf{,}\ T \vdash_{\mathsf{TAL}} \Delta$ (invDistr), $\Gamma[\alpha] \vdash_{\mathsf{TAL}} \Delta$ (classic cut). Hence $\alpha \leqq_{\mathsf{TAL}} T$. Since α is arbitrary, this proves both claims. ⊣

Note that this entails that the relations $\leqq_{\mathsf{TAL}}, \equiv_{\mathsf{TAL}}$ are *not closed under u-substitution*! We will discuss this issue later in more detail. Note also that some additional laws, like the law of semi-complementation or the orthomodular law (well-known from quantum logic), would entail that $Cong(\mathsf{TAL})$ is Boolean, hence they cannot hold.

4.3 On the Congruence Algebra of DAL

In the congruence algebra of DAL, at first glimpse the issue is straightforward: since DAL is closed under e-substitution, we have

$$\alpha \leqq_{\mathsf{DAL}} \beta \text{ if and only if } \alpha \vdash_{\mathsf{DAL}} \beta$$

(only if because of Lemma 8, if because of the rule (cut)). In DAL, and more generally in every distrustful logic L which contains the identity relation on formulas, the outer logic $\vdash_L$ and the inner logic $\leqq_L$ coincide.

There is another distinction which gains some importance, namely between **truth** and **validity** of (in)equations: an equation in $Cong(\mathsf{DAL})$ is valid, if it is true for all uniform substitutions. For example: all *classical* theorems are equivalent in DAL. But the equation

(25) $p \vee \neg p \equiv_{\mathsf{DAL}} q \vee \neg q$

is **true**, not **valid**, since its truth is not preserved by substitutions. Note that the situation is not different from $\leqq_{\mathsf{TAL}}$ (and $\leqq_{\mathsf{cTAL}}$). Lemma 17 entails that the ambiguous substitution of a classical theorem in the truth-theoretic approach can be either strictly true, or undefined, but it is never strictly false. This holds dually for contradictions, so for example $(p\|q) \wedge \neg(p\|q) \vdash_{\mathsf{DAL}} (r\|s) \vee \neg(r\|s)$ holds, as is easy to verify. Hence the inequation

(26) $p \wedge \neg p \leqq_{\mathsf{DAL}} q \vee \neg q$

is **valid** in DAL (as it is in TAL!). As another example, $\alpha \leqq_{\mathsf{DAL}} T$ is true for an arbitrary α and classical theorem T, but $\alpha \leqq T$ is not generally valid in DAL: $p \vdash (p\|q) \vee \neg(p\|q)$ does not hold in DAL: assume p is true, q is false, this falsifies the sequent. Hence there is an even smaller *inner logic of validity*, which satisfies for example the DeMorgan laws, but not (25). That gives rise to a question: are there rules/techniques to make sure that an inequality is *valid* in DAL (not only true)?

Lemma 42 *Assume α, β are positive Boolean formulas, and $\alpha \vdash_{\mathsf{DAL}} \beta$. Then $\alpha \leqq_{\mathsf{DAL}} \beta$ is valid.*

Proof. If there is no ambiguity and no negative polarity involved, then $\blacksquare\sigma(\alpha) = \sigma'(\alpha)$, where for all $p \in Var$, $\sigma'(p) = \blacksquare\sigma(p)$. So assume $\alpha \vdash_{\mathsf{DAL}} \beta$, this entails $\alpha \vdash_{\mathsf{CL}} \beta$, hence $\sigma'(\alpha) \vdash_{\mathsf{CL}} \sigma'(\beta)$, so $\blacksquare\sigma(\alpha) = \sigma'(\alpha) \vdash_{\mathsf{DAL}} \sigma'(\beta) = \blacksquare\sigma(\beta)$. Same for $\blacklozenge$. $\dashv$

This entails a number of things, in particular that all distributive laws for $\wedge, \vee$ are valid in DAL. Moreover, we have double negation and DeMorgan laws:

Lemma 43 *For all formulas α, β we have 1. $\alpha \equiv_{\mathsf{DAL}} \neg\neg\alpha$, 2. $\neg(\alpha \vee \beta) \equiv_{\mathsf{DAL}} \neg\alpha \wedge \neg\beta$, and 3. $\neg(\alpha \wedge \beta) \equiv_{\mathsf{DAL}} \neg\alpha \vee \neg\beta$.*

Proof. 1. We have $\blacksquare(\neg\neg\alpha) \equiv_{\mathsf{CL}} \blacksquare(\alpha)$ for all α, hence also for $\sigma(\alpha)$ for arbitrary σ.

2. $\blacksquare\neg(\alpha \vee \beta) = \neg(\blacklozenge\alpha \vee \blacklozenge\beta) \equiv_{\mathsf{CL}} \neg(\blacklozenge\alpha) \wedge \neg(\blacklozenge\beta) = \blacksquare(\neg\alpha \wedge \neg\beta)$. Dual for $\blacklozenge$.

3. Parallel. $\dashv$

There is a pattern: the important thing is that all formulas occurring multiple times in the equation have the same polarity. Hence every equality, which 1. is valid in Boolean algebras, in which 2. we have the same variables on the left and the right side, and 3. all these variables have the same polarity is valid in the congruence algebra of DAL. Even without this, it is easy enough to prove that *Cong*(DAL) satisfies all requirements for ambiguity like Unambiguous Entailments, Universal Distribution etc.

4.4 On the Congruence Algebra of cTAL

The Context Lemma 30 does obviously also hold for cTAL. Hence we can easily extend the results of Lemma 32 to cTAL. We will not spell them out here, as the presentation would be redundant. The relation $\leqq_{\mathsf{cTAL}}$ has another pleasant surprise for us, as can be seen in the following lemma. Recall that $\alpha \vdash_{\mathsf{cTAL}} \beta$ iff $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$.

Lemma 44 *$\alpha \leqq_{\mathsf{cTAL}} \beta$ iff $\alpha \vdash_{\mathsf{DAL}} \beta$.*

Proof. *If* Assume $\alpha \vdash_{\mathsf{DAL}} \beta$. Hence $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$. Assume moreover $\Gamma[\beta] \vdash_{\mathsf{cTAL}} \Delta$, which means $\blacksquare\Gamma[\beta] \vdash_{\mathsf{CL}} \blacklozenge\Delta$. By (cut) (admissible in classical logic), we then have $\blacksquare\Gamma[\alpha] \vdash_{\mathsf{CL}} \blacklozenge\Delta$, so $\blacksquare\Gamma[\alpha] \vdash_{\mathsf{cTAL}} \blacklozenge\Delta$. Dual argument for $\Gamma \vdash_{\mathsf{cTAL}} \Delta[\alpha]$.

Only if Contraposition: assume $\Vdash\!\!\!\!/_{\mathsf{DAL}}\ \alpha \vdash \beta$. Case 1: $\Vdash\!\!\!\!/_{\mathsf{CL}}\ \blacksquare\alpha \vdash \blacksquare\beta$. This entails by Lemma 24 that $\Vdash\!\!\!\!/_{\mathsf{cTAL}}\ \alpha \vdash \blacksquare\beta$. However, we obviously have $\beta \vdash_{\mathsf{cTAL}} \blacksquare\beta$, so $\alpha \nleqq_{\mathsf{cTAL}} \beta$. Case 2: $\Vdash\!\!\!\!/_{\mathsf{CL}}\ \blacklozenge\alpha \vdash \blacklozenge\beta$. Dually. $\dashv$

This is a simple, yet surprising result: DAL is the **inner logic** of cTAL. Hence $\leqq_{\mathsf{cTAL}}$ is *not* closed under u-substitution, but under e-substitution. This gives us the following interesting corollary: by definition, every outer logic $\vdash_L$ gives rise to a unique inner logic $\leqq_L$. But the converse is not true:

Corollary 45 *There are logics L, L' such that $\vdash_L \neq \vdash_{L'}$, yet $\leqq_L = \leqq_{L'}$*

Hence constructing logics exclusively from their inner logic is a problematic exercise (this will be important in section 5). Let L be an ambiguity logic. Its inner logic $\leqq_L$ is always closed under e-substitution. This leads to the following question: If L is a trustful ambiguity logic, is $\leqq_L$ then a distrustful ambiguity logic? To discuss this question, we need a formal definition of ambiguity logics. This is provided in Definition 49, so if the reader wants to fully understand the following definition and lemma, she needs to have a glimpse at the next page before. For every ambiguity logics L, it is obvious that $\leqq_L$ satisfies (UD),(UE),(id) etc. However, one condition is not necessarily satisfied, namely that $\leqq_L$ extends classical logic.

Definition 46 *We say a trustful ambiguity logic L is* ***classically congruent*** *if $\vdash_{\mathsf{CL}} \subseteq \leqq_L$.*

This gives us another way of constructing distrustful ambiguity logics: namely as *inner logics* of trustful logics – presupposing that their inner logic contains $\vdash_{\mathsf{CL}}$. We think that classical congruence is a central criterion for a trustful logic to be reasonable; we will see some strange logics not satisfying this in the following section.

Lemma 47 *(In-Out Lemma) Assume L is a trustful classically congruent ambiguity logic (in the sense of Definition 49). Then $\leqq_L$ is the consequence relation of a distrustful ambiguity logic.*

Corollary 48 $\leqq_{\mathsf{TAL}}$ *is a non-commutative distrustful ambiguity logic.*

This follows from Lemma 34. Moreover, since $\vdash_{\mathsf{TAL}}$ contains the identity relation of formulas (for all α, $\alpha \vdash_{\mathsf{TAL}} \alpha$), it is straightforward that $\leqq_{\mathsf{TAL}}$ is non-trivial (otherwise TAL would be also m-trivial). Since $\leqq_{\mathsf{TAL}}$ by definition is closed under e-substitution, it cannot be closed under u-substitution. Hence we have another ambiguity logic basically "for free", with the only problem that we do not know very much about it yet. We conjecture the following:

Conjecture 1 $\alpha \leqq_{\mathsf{TAL}} \beta$ *iff* $\alpha \vdash_{\bar{\mathsf{c}}\mathsf{DAL}} \beta$

This would result in a very nice symmetry, but given that both TAL and $\bar{\mathsf{c}}\mathsf{DAL}$ for now only have proof-theoretic characterizations, we expect the proof to be lengthy. Note at this place that TAL cannot be axiomatized by its congruence algebra: there is no set of (in-)equations $\mathbf{E}$ such that TAL is the smallest ambiguity logic satisfying $\mathbf{E}$. The reason for this is, shortly, that $\leqq_{\mathsf{TAL}}$, the inner logic of $\vdash_{\mathsf{TAL}}$, is also a (distrustful) ambiguity logic itself, satisfying obviously exactly the same equations.

We have stated many results on ambiguity logics, but we have not yet formally defined what *is* an ambiguity logic, because we wanted to introduce the key concepts first. In particular, given the concepts in 2.2, should we characterize ambiguity logics in terms of $\vdash$ or $\leqq$ (since we have seen the two do not coincide)? This is what we address in the next section.

5 The Family of Ambiguity Logics

5.1 Meet the Family

We now provide a formal definition of ambiguity logics, which we believe is the most reasonable and natural one. To explain our choice: we have decided that all ambiguity logics have to satisfy the mandatory properties of 2.2 for $\leqq$ and $\equiv$, hence up to congruence. As regards classical connectives, we do not have any requirements except the conservative extension of classical logic. This opens the way to logics where classical operators behave non-classical in ambiguous contexts (see section 5.3). So we want to be strict in the criteria of ambiguity which have to be satisfied, but for the rest be as liberal as possible.

Definition 49 *A logic $L = (Form(\mathsf{AL}), \vdash_L)$ is an **ambiguity logic** if it satisfies the following:*

1. $\vdash_{\mathsf{CL}} \subseteq \vdash_L$ *(extends classical logic)*
2. *L satisfies universal distribution and unambiguous entailments for congruence:*
   (a) $\alpha \wedge (\beta \| \gamma) \equiv_L (\alpha \wedge \beta) \| (\alpha \wedge \gamma)$
   (b) $\alpha \vee (\beta \| \gamma) \equiv_L (\alpha \vee \beta) \| (\alpha \vee \gamma)$
   (c) $\neg(\alpha \| \beta) \equiv_L \neg\alpha \| \neg\beta$
   (d) $\alpha \wedge \beta \leqq_L \alpha \| \beta \leqq_L \alpha \vee \beta$
3. $(\alpha \| \beta) \| \gamma \equiv_L \alpha \| (\beta \| \gamma)$
4. $\alpha \| \alpha \equiv_L \alpha$
5. $\Gamma \vdash_L \Delta$, $\sigma : Var \to Form(\mathsf{CL})$ *entails* $\sigma(\Gamma) \vdash_L \sigma(\Delta)$
6. $\Gamma \vdash_L \alpha$, $\alpha \vdash_L \Delta$, $\alpha \in Form(\mathsf{CL})$ *entails* $\Gamma \vdash_L \Delta$

A note on the list of axioms: properties (1-4) are obvious and have been established as mandatory properties of ambiguity. We do not require the usual properties of abstract logics in the sense of [13], in particular transitivity and closure under u-substitution. But importantly, both properties are only abandoned if ambiguous propositions are involved. Classical propositions should behave classically. Hence 5 states that all classical propositions behave equally, and we refer to functions $\sigma : Var \to Form(\mathsf{CL})$ as classical substitutions. Similarly, 6 states that classical propositions allow for transitive inferences.[13]

Note that 4 is derivable provided that the congruence algebra of L is lattice ordered, in particular if $\alpha \wedge \alpha \equiv_L \alpha \equiv_L \alpha \wedge \alpha$, but this need not generally hold (see (39), which shows that it is independent). Of course, this is a rather minimal choice for properties: we do *not* require that $\wedge$ be associative, idempotent, we do not require the distributive and DeMorgan laws to hold etc. In fact, we will

[13]We will not use condition 6 in this article, but we think it should be included nonetheless.

observe logics without some of these properties. This liberal definition allows us to explore the space of ambiguity logics more freely.

Logics are usually ordered by (set-theoretic) inclusion of $\vdash_L$. We will do the same here, and write $L \subseteq L'$, if $\vdash_L \subseteq \vdash_{L'}$. Note that there is an alternative ordering by the inner logic $\leqq_L$, which of course does not coincide. The reasons for ordering by $\vdash_L$ are rather obvious: firstly, it is way easier to establish inclusion for $\vdash_L$. Secondly and more importantly, distinct logics can have the same inner logic, see Lemma 44. Hence ordering by inner logics would not be very helpful, as distinct logics coincide. Having said this, we can have a closer look at the structure of the family. We write

$$\alpha \leqq_{\mathsf{AL}} \beta \text{ iff } \alpha \leqq_L \beta \text{ for } \textbf{every} \text{ ambiguity logic } L\text{; same for } \equiv_{\mathsf{AL}}$$

Lemma 50 *Assume I is an arbitrary index set, and for all $i \in I$, L_i is an ambiguity logic. Then $\bigwedge\{L_i : i \in I\} = (Form(\mathsf{AL}), \bigcap\{\vdash_i : i \in I\})$ is an ambiguity logic.*

Proof. 1. If $\vdash_{\mathsf{CL}} \subseteq \vdash_i$ for all $i \in I$, then $\vdash_{\mathsf{CL}} \subseteq \bigcap\{\vdash_i : i \in I\}$. 2.-4. It is well-known that arbitrary intersections preserve validity of in-equations. 5.-6. These requirements also have the form of implications, which are preserved under arbitrary intersections. ⊣

By this, we obtain the following:

Lemma 51 *The family of ambiguity logics is a complete, bounded lattice.*

Proof. We have arbitrary intersections, hence we can define $\bigvee\{L_i : i \in I\} = \bigwedge\{L_j : \bigcup\{\vdash_i : i \in I\} \subseteq \vdash_j\}$ (unions do not preserve certain properties, in particular 6).

Its maximal element is obviously the inconsistent logic $Form(\mathsf{AL}) \times Form(\mathsf{AL})$ (which coincides with cTAL^{cut}). Its minimal element exists by completeness; it is the logic $L_{\blacklozenge\blacksquare}$ at which we will have a closer look below. ⊣

Note that completeness is important for extensions: We can close a distrustful logic L under u-substitution. The result will not necessarily be an ambiguity logic, but we can intersect all ambiguity logics which contain this substitution closure, which we then call the trust closure of L (see section 5.3).

5.2 More Logics of Distrust: van Deemter's Approach

We have seen two logics based on/related to the "truth operators" $\blacksquare, \blacklozenge$:

- We have $\alpha \vdash_{\mathsf{DAL}} \beta$ iff $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$ and $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$
- We have $\alpha \vdash_{\mathsf{cTAL}} \beta$ iff $\blacksquare\alpha \vdash \blacklozenge\beta$

It is actually very easy to construct further ambiguity logics with these operators: recall Lemma 22, which states that if $\gamma \in anf(\alpha)$, then $\blacksquare\gamma \equiv_{\mathsf{CL}} \blacksquare\alpha$, and $\blacklozenge\gamma \equiv_{\mathsf{CL}} \blacklozenge\alpha$. Moreover, the following is straightforward:

Lemma 52 *1.* $\blacksquare(\alpha \wedge \beta) \vdash_{\mathsf{CL}} \blacksquare(\alpha \| \beta) \vdash_{\mathsf{CL}} \blacksquare(\alpha \vee \beta)$

2. $\blacklozenge(\alpha \wedge \beta) \vdash_{\mathsf{CL}} \blacklozenge(\alpha \| \beta) \vdash_{\mathsf{CL}} \blacklozenge(\alpha \vee \beta)$

These and some other simple results ensure that whenever we use the operators $\blacksquare$, $\blacklozenge$ to define a consequence relation in terms of classical logic for the language of ambiguity logics, we can be sure that it actually is an ambiguity logic. Hence we can easily construct the commutative ambiguity logics $L_{\blacklozenge\blacklozenge}, L_{\blacksquare\blacksquare}, L_{\blacklozenge\blacksquare}$. To avoid double subscripts, we write $\vdash_{\blacklozenge\blacklozenge}$ instead of $\vdash_{L_{\blacklozenge\blacklozenge}}$ etc.

1. $L_{\blacksquare\blacksquare}$: $\alpha \vdash_{\blacksquare\blacksquare} \beta$ iff $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$
2. $L_{\blacklozenge\blacklozenge}$: $\alpha \vdash_{\blacklozenge\blacklozenge} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$
3. $L_{\blacklozenge\blacksquare}$: $\alpha \vdash_{\blacklozenge\blacksquare} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$

We will quickly discuss these ambiguity logics here. They have, in a different setting and different form, been introduced in [15].

$L_{\blacksquare\blacksquare}$ and $L_{\blacklozenge\blacklozenge}$ These two logics are dual, so we can treat them in one. The previous lemmas already entail that these are ambiguity logics. Obviously,

(27) $\vdash_{\mathsf{DAL}} = \vdash_{\blacksquare\blacksquare} \cap \vdash_{\blacklozenge\blacklozenge}$

What is the difference between the two logics? We have

(28) $$p \vee q \equiv_{\blacklozenge\blacklozenge} p\|q$$

(29) $$p \wedge q \not\equiv_{\blacklozenge\blacklozenge} p\|q$$

(30) $$p \vee q \not\equiv_{\blacksquare\blacksquare} p\|q$$

(31) $$p \wedge q \equiv_{\blacksquare\blacksquare} p\|q$$

Hence in both logics '$\|$' coincides with a classical connective. Note however that the logics are not trivially equivalent to classical logic; we illustrate this for $L_{\blacklozenge\blacklozenge}$:

(32) $\neg(p\|q) \equiv_{\blacklozenge\blacklozenge} \blacklozenge\neg(p\|q) \equiv_{\blacklozenge\blacklozenge} \neg\blacksquare(p\|q) \equiv_{\blacklozenge\blacklozenge} \neg(p \wedge q) \equiv_{\blacklozenge\blacklozenge} \neg p \vee \neg q$

We also obviously have $p \vee q \vdash_{\blacklozenge\blacklozenge} p\|q$, yet

(33) $\neg(p\|q) \not\vdash_{\blacklozenge\blacklozenge} \neg(p \vee q)$

This means that the rule of contraposition is not sound in $L_{\blacklozenge\blacklozenge}$ (and $L_{\blacksquare\blacksquare}$), contrary to all logics we have observed until now. In our view, $L_{\blacklozenge\blacklozenge}, L_{\blacksquare\blacksquare}$ are two logics which

1. derive undesirable sequents, and
2. do not allow for intuitive inference rules.

Hence they do not seem to be particularly interesting.

$L_{\blacklozenge\blacksquare}$, the minimal ambiguity logic $L_{\blacklozenge\blacksquare}$ is a very particular and restrictive logic. In particular, $\nVdash_{\blacklozenge\blacksquare} a\|b \vdash a\|b$. This means that (I**;** I) is not sound in this logic. This provides us also with an example of a logic where ($\|$mon) does not hold. It is easy to see that $L_{\blacklozenge\blacksquare}$ is a logic of *extreme mistrust*:

$$a_1\|...\|a_i \vdash_{\blacklozenge\blacksquare} b_1\|...\|b_j$$

is derivable iff under an *arbitrary* left reading a_n, *all* right readings b_m follow. It is easy to see that $L_{\blacklozenge\blacksquare}$ is closed under e-substitution, yet not under u-substitution. We can actually prove that $L_{\blacklozenge\blacksquare}$ is the minimal ambiguity logic:

Lemma 53 *$L_{\blacklozenge\blacksquare}$ is the smallest ambiguity logic.*

Proof. Assume without loss of generality that $\gamma = a_1\|...\|a_n$, $\delta = b_1\|...\|b_m$ are both in ambiguous normal form. Assume moreover $\blacklozenge\gamma \vdash_{\mathsf{CL}} \blacksquare\delta$. Hence for an arbitrary ambiguity logic L, we have $\blacklozenge\gamma \vdash_L \blacksquare\delta$ (by conservative extension). Then, since $\alpha\|\beta \leqq_L \alpha \vee \beta$, we obtain $\gamma \vdash_L \blacksquare\delta$, and since $\alpha \wedge \beta \leqq_L \alpha\|\beta$, we obtain $\gamma \vdash_L \delta$. ⊣

Hence even if $L_{\blacklozenge\blacksquare}$ does not seem to be very useful for many purposes, it has a special position in the family of ambiguity logics. Note a peculiarity: $L_{\blacklozenge\blacksquare}$ is *commutative*. Yet, it is not only minimal for commutative logics, but for all ambiguity logics. Hence the smallest ambiguity logic is commutative, and every non-commutative ambiguity logic contains it. Finally, note that $L_{\blacklozenge\blacksquare}$ does *not* even satisfy the weak law of disambiguation:

(34) $(p\|q\|r) \wedge \neg q \vdash p\|r$

is *not* valid in $L_{\blacklozenge\blacksquare}$ (easy to check). This is directly connected to the fact that (I**;** I) is not sound in this logic.

$L_{\blacklozenge\blacksquare}$ provides us also with an example of a logic where $\leqq_{\blacklozenge\blacksquare} \not\subseteq \vdash_{\blacklozenge\blacksquare}$, that is: the outer logic does not contain the inner logic: we do not have $\alpha \vdash_{\blacklozenge\blacksquare} \alpha$ for all α, but by definition $\alpha \leqq_{\blacklozenge\blacksquare} \alpha$ (see also Lemma 8). The inner logic $\leqq_{\blacklozenge\blacksquare}$ is in this case actually *larger* than the outer logic. What can we say about the inner logic $\leqq_{\blacklozenge\blacksquare}$? A quite surprising lemma is the following:

Lemma 54 *$\alpha \leqq_{\blacklozenge\blacksquare} \beta$ if and only if $\alpha \vdash_{\mathsf{DAL}} \beta$.*

Proof. *If*: Assume $\alpha \vdash_{\mathsf{DAL}} \beta$. Hence $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$, $\blacklozenge\alpha \vdash \blacklozenge\beta$. Assume $\Gamma[\beta] \vdash_{\blacklozenge\blacksquare} \Delta$, so $\blacklozenge\Gamma[\beta] \vdash_{\mathsf{CL}} \blacksquare\Delta$. This entails that $\blacklozenge\Gamma[\alpha] \vdash_{\mathsf{CL}} \blacksquare\Delta$. Dually on the right.

Only if: Assume $\alpha \nvdash_{\mathsf{DAL}} \beta$. Case 1: $\blacksquare\alpha \nvdash_{\mathsf{CL}} \blacksquare\beta$. We have $\blacksquare\alpha \vdash_{\blacklozenge\blacksquare} \alpha$, but by assumption $\blacksquare\alpha \nvdash_{\blacklozenge\blacksquare} \beta$, and hence $\alpha \nleqq_{\blacklozenge\blacksquare} \beta$. Case 2: $\blacklozenge\alpha \nvdash_{\mathsf{CL}} \blacklozenge\beta$: dually. ⊣

This is remarkable: cTAL, a trivial trustful ambiguity logic, and the smallest ambiguity logic (which is distrustful) have the same identical inner logic, which is DAL!

5.3 Trust Closure and the Trust Theorem

5.3.1 The Trust Theorem

We have seen logics of trust and logics of distrust. The Trust Theorem is crucial for understanding their relation. Assume L_t is a logic of trust, L_d a logic of distrust. This obviously does not entail that $L_d \subseteq L_t$: just consider that $\mathsf{DAL} \not\subseteq \mathsf{TAL}$, because of commutativity. But there is one important question: is there a trustful logic L_t, distrustful logic L_d, such that $L_t \subseteq L_d$? This would somehow contradict our intuitive conception of what trust means: if we trust each other, we tend to accept additional valid arguments and inferences, which we would refute if we distrust (see (14),(6)) – otherwise what would be the point of trust? We will prove that trust always increases the set of valid inferences. The underlying reason for this is that e-substitution preserves closure under u-substitution, but not vice versa. Assume L is an ambiguity logic. We let eL, its closure under e-substitution, denote the smallest logic such that

1. $\Gamma \vdash_L \Delta$ entails $\Gamma \vdash_{eL} \Delta$
2. $\Gamma[\alpha] \vdash_{eL} \Delta[\beta]$, $\alpha' \vdash_{eL} \alpha$, $\beta \vdash_{eL} \beta'$ entail $\Gamma[\alpha'] \vdash_L \Delta[\beta']$

Though e is not a function, it behaves as a closure operator: $L \subseteq eL$, if $L \subseteq L'$, then $eL \subseteq eL'$, and $e(eL) = eL$.

Lemma 55 *For every trustful ambiguity logic L, $\leqq_L \subseteq \leqq_{eL}$*

Proof. In every trustful ambiguity logic L, we have $\gamma \vdash_L \gamma$ for all $\gamma \in$ *Form*(AL), and hence $\leqq_L \subseteq \vdash_L$ (Lemma 8). Assume $\alpha \leqq_L \beta$. Then $\alpha \vdash_L \beta$, hence $\Gamma[\beta] \vdash_{eL} \Delta$ entails $\Gamma[\alpha] \vdash_{eL} \Delta$. Same on the right, so $\alpha \leqq_{eL} \beta$ $\dashv$

Lemma 56 *Assume L is a trustful ambiguity logic. Then eL is a trustful ambiguity logic as well.*

Proof. eL is an ambiguity logic, since we firstly have $\vdash_L \subseteq \vdash_{eL}$ and secondly $\leqq_L \subseteq \leqq_{eL}$, so we satisfy conditions 1.-4. 6. is immediate. To see that eL is closed under u-substitution: L is trustful, so $\Gamma[\alpha] \vdash_L \Delta[\beta]$, $\alpha' \vdash_L \alpha$, $\beta \vdash_L \beta'$ entail $\sigma(\Gamma[\alpha]) \vdash_L \sigma(\Delta[\beta]$, $\sigma(\alpha') \vdash_L \sigma(alpha)$ etc. Hence $\sigma(\Gamma[\alpha']) \vdash_{eL} \sigma(\Delta[\beta'])$. More formally, closure under σ is preserved over substitution steps. $\dashv$

Note also that if L is distrustful, then $L = eL$. This entails that an ambiguity logic of trust cannot be extended to an ambiguity logic of distrust:

Lemma 57 *Assume L_d is a non-trivial distrustful ambiguity logic, L_t a trustful ambiguity logic. Then $\vdash_{L_t} \not\subseteq \vdash_{L_d}$.*

Proof. Assume $\vdash_{L_t} \subseteq \vdash_{L_d}$. Then we have $\vdash_{L_t} \subseteq \vdash_{eL_t} \subseteq \vdash_{eL_d} = \vdash_{L_d}$. However, eL_t is by the previous lemma closed under both e- and u-substitution and hence m-trivial (Fundamental Theorem). So L_d has to be m-trivial too, contradiction. $\dashv$

This result confirms our intuitive approach: there is an asymmetry between e-substitution and u-substitution, which entails that under trustful reasoning, we have more valid inferences than under distrustful reasoning. Now we consider the other direction. We first define an intermediate concept, namely **substitution closure**: we define σL by $\alpha \vdash_{\sigma L} \beta$ if there are α', β' and a u-substitution σ such that $\alpha' \vdash_L \beta'$, $\sigma(\alpha') = \alpha$, $\sigma(\beta') = \beta$. Note that σL need not be an ambiguity logic in the sense of Definition 49!

Definition 58 *For an ambiguity logic L, τL, the **trust closure** of L, is the smallest ambiguity logic containing σL.*

Completeness of the lattice of ambiguity logics already entails that every ambiguity logic L is included in a unique smallest trustful ambiguity logic τL. Of course,

(35) $\vdash_L \subseteq \vdash_{\sigma L} \subseteq \vdash_{\tau L}$

We can alternatively define the trust closure τL as the smallest logic which 1. contains L, 2. is closed under u-substitution, and 3. is closed under all conditions in Definition 49. The most important question is: does trust-closure preserve non-triviality? We will settle this question positively, then show two examples of trust closure. In the following proofs, we will often use formulas of the form $p_1\|...\|p_n$, that is, with atoms and $\|$ only. We will abbreviate these as words $p_1...p_n$. This allows us to use conventions from language theory, like p^n, denoting $p\|p\|...\|p$ (n times). We will abbreviate this type of formula with variables w, v, u. Recall that the notion of m-triviality (Definition 4) is firstly not defined over $\leqq$ but only over $\vdash$, and secondly, it is not only defined for ambiguity logics, but for all logics in the language.

Lemma 59 *Assume L is an ambiguity logic. Then L is trivial if and only if σL is trivial.*

Proof. *Only if* is obvious by inclusion.

If: Assume σL is trivial, hence $pqr \vdash_{\sigma L} pq'r$ for all $p, q, q', r \in \mathit{Var}$. Pick arbitrary atoms where all four are pairwise distinct. We now distinguish two cases:

Case 1: $pqr \nvdash_L pq'r$. Hence the formula must be the result of a substitution. Now there are several different subcases (i.e. pre-images), for example $ps \vdash_L pq'r$. In this case, we classically substitute $s \mapsto q \vee r$, so $p(q \vee r) \vdash_L pq'r$, hence $pqr \vdash_L pq'r$ – contradiction. Similar for all other possible pre-images of the sequent.

Case 2: $pqr \vdash_L pq'r$. Then take an arbitrary sequent $\alpha\|\beta\|\gamma \vdash \alpha\|\beta'\|\gamma$, where $\alpha\|\beta\|\gamma$, $\alpha\|\beta'\|\gamma$ are in ambiguous normal form, and $\alpha, \gamma \in \mathit{Form}(\mathsf{CL})$. Put

$$\sigma : p \mapsto \alpha,\ \sigma : q \mapsto \blacklozenge\beta,\ \sigma : r \mapsto \gamma,\ \sigma : q' \mapsto \blacksquare\beta'.$$

Since σ is classical, we have $\alpha\|\blacklozenge\beta\|\gamma \vdash_L \alpha\|\blacksquare\beta'\|\gamma$. So by (UE) we have $\alpha\|\beta\|\gamma \vdash_L \alpha\|\beta'\|\gamma$, so we can derive arbitrary ambiguous normal forms, and by (UD), it follows that $\alpha\|\beta\|\gamma \vdash_L \alpha\|\beta'\|\gamma$ for arbitrary $\alpha, \beta, \beta', \gamma$. Hence L is trivial. ⊣

Observation 2 *By definition of* τL*,* $\alpha \vdash_{\tau L} \beta$ *iff there exists* α', β' *such that* $\alpha \leqq_{\mathsf{AL}} \alpha' \vdash_{\sigma L} \beta' \leqq_{\mathsf{AL}} \beta$*. This makes the following interesting:*

Observation 3 *Assume* $p, q, r \in Var$*, and either* $pqr \leqq_{\mathsf{AL}} p_1...p_i$ *or* $p_1...p_i \leqq_{\mathsf{AL}} pqr$*. Then* $p_1...p_i = p^m q^n r^o$*. This is obvious, since the (id) axiom is the only applicable axiom.*

Lemma 60 *Assume* $\sigma(w) = p^m q^n r^o$*, where* $p, q, r \in Var$ *are pairwise distinct,* $m, n, o > 0$*. Then* $\sigma = \sigma_1 \circ \sigma_2$*, where* σ_2 *is classical, and* $\sigma_2(w) = (p_1)^{n_1}...(p_i)^{n_i}$*, such that if* $j \neq k$*, then* $p_j \neq p_k$*.*

Proof. Assume $\sigma((p_1)^{n_1}...(p_i)^{n_i}) = p^m q^n r^o$, $p_j = p_{j+k}$, where $k > 1$ (if $k = 1$, then $(p_j)^{n_j}(p_{j+1})^{n_{j+1}} = (p_j)^{n_j + n_{j+1}}$). Now there are two cases:

Case 1: there is $n' : n < n' < n + k$ such that $\sigma(p_{n'}) = x^a$, $\sigma(p_n) = y^b$, $x \neq y$. In this case $p^m q^n r^o = v y^a x^b y^c u$ with $x \neq y$ – contradiction.

Case 2: This n' does not exist. Then we put $\sigma_2(p_{n'}) = p_n$, $\sigma_1 = \sigma$. This ensures $\sigma = \sigma_1 \circ \sigma_2$, and σ_2 is classical. ⊣

Lemma 61 *Let* σ *be a substitution such that* $\sigma(w) = p^m q^n r^o$*,* $\sigma(v) = p^{m'} q'^{n'} r^{o'}$*, where* $p, q, q', r \in Var$*, and* $w \not\equiv_{\mathsf{AL}} v$*. Then there is a classical substitution* $\sigma' : Var \to Form(\mathsf{CL})$ *such that*

1. *for all variables* s *occurring in* w*,* $\sigma(s) \leqq_{\mathsf{AL}} \sigma'(s)$
2. *for all variables* s *occurring in* v*,* $\sigma'(s) \leqq_{\mathsf{AL}} \sigma(s)$

Note that in particular if $w \vdash_L v$, then $p^m q^n r^o \leqq_{\mathsf{AL}} \sigma'(w)$ and $\sigma'(v) \leqq_{\mathsf{AL}} p^{n'} q'^{n'} r^{o'}$ entails that $p^m q^n r^o \vdash_L p^{n'} q'^{n'} r^{o'}$, since σ' is classical.

Proof. Induction over $|wv|$. Basis: 2. Then $w = s$, $v = s'$ and $s \neq s'$. Then put $\sigma'(s) = \blacklozenge(p^m q^n r^o)$, $\sigma'(s') = \blacksquare(p^{n'} q'^{n'} r^{o'})$, which satisfies the claim. As IH, assume the claim holds for all w, v with $|wv| \leq n$, and we have $sw \vdash_L v$, $|swv| = n + 1$.

Case 1: s does occur in w. By IH $\sigma(s) \leqq_{\mathsf{AL}} \sigma'(s)$, and since $\sigma(sw) = p^m q^n r^o$, s occurs in w, $\sigma(s) = p^i$. We can simply put $\sigma'(s) = p$; this satisfies the claim.

Case 2. s does occur in v. Then we have $\sigma'(s) \leqq_{\mathsf{AL}} \sigma(s)$. But since s occurs both on left and right of $\vdash$, we must have $\sigma(s) = p^i$. So we can simply put $\sigma'(s) = p$, which is classical and $\equiv_{\mathsf{AL}}$-equivalent.

Case 3. s does not occur in wv. Then just put $\sigma'(s) = \blacklozenge\sigma(s)$, this satisfies the claim.

Similar for $ws \vdash_L v$, $w \vdash_L sv$, $w \vdash_L vs$. ⊣

Lemma 62 *Assume* σL *is non-trivial. Then* τL *is non-trivial.*

Proof. Assume τL is trivial, hence $pqr \vdash_{\tau L} pq'r$, where $p, q, q', r \in Var$ are pairwise distinct. This means that there are α, β such that $pqr \leqq_{\mathsf{AL}} \alpha \vdash_{\sigma L} \beta \leqq_{\mathsf{AL}} pq'r$. Now since $pqr \leqq_{\mathsf{AL}} \alpha$, α can only contain $\|, \vee$, and dually β can only contain $\|, \wedge$. $\alpha \vdash_{\sigma L} \beta$ means we have $\alpha' \vdash_L \beta'$ and a substitution σ, such that $\sigma(\alpha') = \alpha$, $\sigma(\beta') = \beta$, where α' contains only connectives $\|, \vee$, β' only $\|, \wedge$ (since substitutions preserve connectives).

Assume $\alpha = \alpha''[\beta \vee \gamma]$. There are two cases:

Case 1: $\vee$ did occur in α' already. Then since L is an ambiguity logic, it can be substituted by $\|$ preserving validity of the sequent.

Case 2: $\beta \vee \gamma$ is the result of a substitution $\sigma(s)$. Since β does not contain $\vee$, β' cannot contain s. Hence we can change $\sigma'(s) = \beta \| \gamma$.

Dually for β, this shows that without loss of generality, we can assume that $\alpha = p^m q^n r^o, \beta = p^{m'} q'^{n'} r^{o'}$ (see Observation 3). Lemma 60 entails that there are $\alpha' = p_1^{n_1} ... p_i^{n_i}$, $\beta' = q_1^{n'_1} ... q_j^{n'_j}$, where $\alpha' \vdash_L \beta'$. By Lemma 61, we then have $\alpha \vdash_L \beta$, and since $pqr \leqq_{\mathsf{AL}} \alpha$, $\beta \leqq_{\mathsf{AL}} pq'r$, we have $pqr \vdash_L pq'r$. Hence σL is trivial. $\dashv$

Corollary 63 *(Trust Theorem)*

1. *Every distrustful ambiguity logic L can be extended to a unique smallest trustful ambiguity logic τL, where $\vdash_L \subseteq \vdash_{\tau L}$. Moreover, if L is non-trivial, then τL is non-trivial.*
2. *No trustful ambiguity logic can be extended to a non-trivial distrustful ambiguity logic: if L_t is a trustful ambiguity logic, L_d is a distrustful ambiguity logic, and $\vdash_{L_t} \subseteq \vdash_{L_d}$, then L_d is trivial.*

In the remainder of this subsection, we will see two examples of trust closure. We will see that trust closures usually have rather odd properties. This will lead to the last method for constructing ambiguity logics, namely the one of algebraic extensions of congruence algebras.

5.3.2 τDAL

Our first example for trust closure is τDAL, the trust closure of DAL. τDAL is the smallest logic such that

1. DAL $\subseteq$ τDAL
2. If $\Gamma \vdash_{\mathsf{DAL}} \Delta$, then $\sigma(\Gamma) \vdash_{\mathsf{DAL}} \sigma(\Delta)$
3. τDAL is an ambiguity logic.

Every sequent in τDAL can be derived from a sequent in DAL via 1. a substitution followed by 2. application of the ambiguity laws in Definition 49 (since they preserve closure under substitution, see Lemma 56). But already DAL satisfies these laws, hence if $\alpha \equiv_{\mathsf{AL}} \alpha'$, then $\vdash_{\mathsf{DAL}} \alpha$ iff $\vdash_{\mathsf{DAL}} \alpha'$, moreover if $\alpha \leqq_{\mathsf{AL}} \alpha'$, then if $\not\vdash_{\mathsf{DAL}} \alpha'$, then $\not\vdash_{\mathsf{DAL}} \alpha$. Same for contradictions. We first show the following lemma, which is very powerful:

Lemma 64 *Assume $p \in$ Var, $\alpha, \beta \in$ Form(CL), where α is not a theorem, β not a contradiction, and p does not occur in α, β. Then $(\beta \| p) \vee \neg(\alpha \| p)$ is a theorem in τDAL iff $\alpha \leqq_{\mathsf{AL}} \beta$.*

Proof. *If* Straightforward by u-substitution: we have $\vdash_{\tau\mathsf{DAL}} (\alpha\|p)\vee\neg(\alpha\|p)$, hence $\vdash_{\tau\mathsf{DAL}} (\alpha\mathbf{;}\ p)\mathbf{,}\ \neg(\alpha\|p)$. If $\alpha \leqq_{\mathsf{AL}} \beta$, it follows that $\vdash_{\tau\mathsf{DAL}} (\beta\mathbf{;}\ p)\mathbf{,}\ \neg(\alpha\|p)$ etc.

Only if Assume $\vdash_{\tau\mathsf{DAL}} (\beta\|p)\vee\neg(\alpha\|p)$, where all premises are satisfied. This means that $\sigma(T) \leqq_{\mathsf{AL}} (\beta\|p)\vee\neg(\alpha\|p)$, where T is a theorem of DAL. Now let us distinguish cases as to the form of T. We distinguish only up to $\equiv_{\mathsf{AL}}$.

i) $T \equiv_{\mathsf{AL}} (\beta'\|p')\vee\neg(\alpha'\|p'')$. Since classical theorems are preserved by classical substitution and $\equiv_{\mathsf{AL}}$-congruence, β' cannot be a theorem. β' can also not contain p'', otherwise β would contain p. Hence there exists $M : M \not\models \beta'$, $M \models p''$. As is easy to check, $M \not\models \blacksquare((\beta'\|p')\vee\neg(\alpha'\|p''))$, so $\not\Vdash_{\mathsf{DAL}} T$ – contradiction.

ii) $T \equiv_{\mathsf{AL}} (\beta'\|p')\vee\neg p''$ In this case, we can take the same model M as in i), which falsifies $\blacksquare((\beta'\|p')\vee\neg p'')$, so $\not\Vdash_{\mathsf{DAL}} T$ – contradiction.

iii) $T \equiv_{\mathsf{AL}} p'\vee\neg(\alpha'\|p'')$ By the dual considerations, α' cannot be a contradiction, and cannot contain p'. Hence there is a model $M : M \models \alpha'$, $M \not\models p'$, hence $M \not\models \blacksquare(p'\vee\neg(\alpha'\|p''))$, so $\not\Vdash_{\mathsf{DAL}} T$ – contradiction.

iv) $T \equiv_{\mathsf{AL}} p'\vee\neg p''$ In this case, we need $p' = p''$, hence $\sigma(T) \equiv_{\mathsf{AL}} (\beta\|p)\vee\neg(\alpha\|p)$, where $\alpha = \beta$. In $\tau\mathsf{DAL}$, no $\leqq_{\mathsf{AL}}$-substitution is applicable to α in this formula, because it is in negative context (and there is no invertible negation rule), and no $\equiv_{\mathsf{AL}}$-substitution is applicable because it is classical (the only one in question would be UD). However, $\leqq_{\mathsf{AL}}$-substitutions might be applied to β (i.e. β is substituted by a $\leqq_{\mathsf{AL}}$-larger formula), and hence $\alpha \leqq_{\mathsf{AL}} \beta$. ⊣

This can be put to use immediately: the formula $(p\|q)\vee\neg(p\|q)$ is obviously a theorem of $\tau\mathsf{DAL}$ ($\sigma(p\vee\neg p)$). However, the formulas in (36–38) are all not theorems of $\tau\mathsf{DAL}$; hence neither double negation nor distributive and DeMorgan laws hold in $\tau\mathsf{DAL}$:

(36) $$((\neg\neg q)\|p) \quad\vee\quad \neg(q\|p)$$

(37) $$((q_1\vee(q_2\wedge q_3))\|p) \quad\vee\quad (\neg(((q_1\vee q_2)\wedge(q_1\vee q_3)))\|p)$$

(38) $$((\neg(q_1\vee q_2))\|p) \quad\vee\quad \neg(((\neg q_1)\wedge(\neg q_2)\|p)$$

are all *not* theorems in $\tau\mathsf{DAL}$. This follows from Lemma 64. Hence we have the following:

Lemma 65 *In* $\tau\mathsf{DAL}$ *we have*

1. $\neg\neg p \not\equiv_{\tau\mathsf{DAL}} p$
2. $\neg(p\wedge q) \not\equiv_{\tau\mathsf{DAL}} \neg p\vee\neg q$
3. $\neg(p\vee q) \not\equiv_{\tau\mathsf{DAL}} \neg p\wedge\neg q$
4. $p\wedge(q\vee r) \not\equiv_{\tau\mathsf{DAL}} (p\wedge q)\vee(p\wedge r)$
5. $p\vee(q\wedge r) \not\equiv_{\tau\mathsf{DAL}} (p\vee q)\wedge(p\vee r)$

Corollary 66 $\tau\mathsf{DAL} \subsetneq \mathsf{cTAL}$

This puts some of our most important results in a nice perspective: we have pointed out in this and in previous publications, that in trustful reasoning with ambiguity, it is important to keep track of some *syntactic* information. We cannot permit full substitution of equivalents, or logics become trivial. Now our results on $\tau\mathsf{DAL}$ indicate: we can choose up to which degree we want to have

Boolean laws in our congruence, that is, up to which degree want to abstract away from syntactic form, where τDAL is a very minimal choice.

This gives us also a means to prove that for example the axiom (id) is independent of the others: assume we would not require closure under (id). Then we still obtain a result similar to Lemma 64, and there are formulas of the form

(39) $(p\|p)\|q \vee \neg(p\|q)$

which would not be theorems in τDAL.

5.3.3 $\tau L_{\blacklozenge\blacksquare}$

Another logic worth looking at is the *smallest trustful ambiguity logic*, which is the trust closure of $L_{\blacklozenge\blacksquare}$, denoted by $\tau L_{\blacklozenge\blacksquare}$, where $L_{\blacklozenge\blacksquare}$ is the minimal ambiguity logic. Since we consider uniform substitution as the mathematical equivalent of "trust", meaning that argument schemes are valid independently of their actual content, then $\tau L_{\blacklozenge\blacksquare}$ is the smallest logic of trust. Note that whereas $L_{\blacklozenge\blacksquare}$ is commutative, $\tau L_{\blacklozenge\blacksquare}$ is not: a formula as $(p\|q) \vee \neg(q\|p)$ is *not* a theorem in $L_{\blacklozenge\blacksquare}$. Many things we have stated in the previous section can be re-stated here. We also have

Lemma 67 $\tau L_{\blacklozenge\blacksquare} \subseteq \mathsf{TAL} \cap \tau\mathsf{DAL}$

This is straightforward. Note that $\tau L_{\blacklozenge\blacksquare}$ is considerably stronger than $L_{\blacklozenge\blacksquare}$. In particular, for all α, we have $\alpha \vdash_{\tau L_{\blacklozenge\blacksquare}} \alpha$, since $p \vdash_{\blacklozenge\blacksquare} p$. Still, (I**;** I) is not sound in this logic: we can derive $p \vdash_{\tau L_{\blacklozenge\blacksquare}} p \vee q$, $r \vdash_{\tau L_{\blacklozenge\blacksquare}} r \vee s$, but

(40) $p\|r \vdash (p \vee q)\|(r \vee s)$

is *not* derivable in $\tau L_{\blacklozenge\blacksquare}$: it cannot be a substitution of any formula derivable in $L_{\blacklozenge\blacksquare}$, and it cannot be obtained by means of distributive laws either, as is simple, but lengthy to check (there is no distributive law applicable to it). This entails that (**,** weak) is not admissible in $\tau L_{\blacklozenge\blacksquare}$, and so $Cong(L_{\blacklozenge\blacksquare})$ is not lattice ordered! Note also the following:

Lemma 68 $\tau L_{\blacklozenge\blacksquare} \not\subseteq \mathsf{DAL}$

This follows as a corollary from the Trust Theorem, similarly $\tau L_{\blacklozenge\blacksquare} \not\subseteq L_{\blacklozenge\blacklozenge}$ etc. A sequent which illustrates this is $\vdash (p\|q) \vee \neg(p\|q)$, valid in $\tau L_{\blacklozenge\blacksquare}$. We also have inverse results:

Lemma 69 $\mathsf{DAL} \not\subseteq \tau L_{\blacklozenge\blacksquare}$

This follows from (40), which is derivable in DAL, yet not in $\tau L_{\blacklozenge\blacksquare}$. Hence the two are incomparable. In general, we think logics lacking the admissibility of (I**;** I) are not very much worth investigating, with the notable exception of $L_{\blacklozenge\blacksquare}$, because of its obvious prominence and simplicity.

5.4 Algebraic Extensions

We have defined the family of ambiguity logics partly "algebraically", that is, by means of (in)equations on the congruences which they satisfy. It is of course tempting to extend this approach and define specific ambiguity logics by additional equations which their congruence algebras have to satisfy, as we define specific algebras from general ones. The results presented so far show that it is not a very promising idea to define logics as "the smallest ambiguity logic L such that $Cong(L)$ satisfies set of equations $\{E_1, ..., E_n\}$". The reason is that the minimal ambiguity logic $L_{\blacklozenge\blacksquare}$ satisfies exactly the same set of equations as our trivial commutative ambiguity logic cTAL. This means: firstly, ambiguity logics are not uniquely determined by their congruence algebra/inner logics, and secondly, the concept of "smallest logic satisfying" is rather useless, since the logics which do not satisfy certain standard congruences are not small, but relatively large with respect to $L_{\blacklozenge\blacksquare}$ (see for example $\tau\mathsf{DAL}$).

A concept which is more useful though, as we have seen for $\tau\mathsf{DAL}$, is the concept of **algebraic extension**. For example, take the logic $\tau\mathsf{DAL}$ and a set of equations $\{E_1, ..., E_n\}$. Then $\tau\mathsf{DAL}(E_1, ..., E_n)$ is the smallest trustful ambiguity logic which which 1. contains DAL and 2. whose congruence algebra satisfies $E_1, ..., E_n$ (meaning they are valid). Note that algebraic extensions are meaningful mostly for trustful logics, since in distrustful logics the inner logic and the outer logic coincide, so we can implement extensions by proof rules. So we define the following:

Definition 70 *Assume L is an ambiguity logic. Then $\tau L(E_1, ..., E_n)$ is the smallest trustful ambiguity logic which contains L and whose congruence algebra satisfies $E_1, ..., E_n$.*

For example in $\tau\mathsf{DAL}$ basically none of the Boolean axioms hold up to congruence. From here, we can construct

- $\tau\mathsf{DAL}(DN)$, which is $\tau\mathsf{DAL}$ with double negation congruence. So the smallest ambiguity logic containing DAL, closed under uniform substitution and satisfying $\neg\neg\alpha \equiv \alpha$
- There is a logic $\tau\mathsf{DAL}(DN, DM)$, which is $\tau\mathsf{DAL}(DN)$ with DeMorgan congruence, etc.

Hence we can add all sorts of congruence-axioms, resulting in a lattice order of logics. Now this opens an interesting question for $\mathsf{TAL}, \mathsf{cTAL}$: can we characterize cTAL as axiomatic extension of $\tau\mathsf{DAL}$, and TAL as extension of $\tau\bar{\mathsf{c}}\mathsf{DAL}$? We have already proved that the congruence algebra $Cong(\mathsf{cTAL})$ is a Kleene algebra. One might ask whether conversely $\tau\mathsf{DAL}(Kleene)$ ($\tau\mathsf{DAL}$ with all axioms for Kleene algebras) is equal to cTAL. The answer is probably negative, the argument is as follows: The rules ($\neg$I),(I$\neg$) preserve $\leqq$ if and only if the negated formula is classical. Since validity of equations means truth under arbitrary substitutions, there is no hope of bringing this into an algebraic form.

Conjecture 2 *The rules* $(\neg I)$*,*$(I\neg)$ *are not sound in* $\tau\mathsf{DAL}(Kleene)$

More generally, the correspondence of proof rules and (in)equations is problematic for a number of reasons.

1. Types of context: a rule such as $(\vee I)$ relates to an inequality $\alpha \leqq \alpha \vee \beta$, because it can be applied in all contexts. However, rules such as $(\neg I)$,$(I\neg)$ or $(\mathbf{I;}\ I)$, which can be applied only in classical context, have no translation into semi-equalities.

2. Properties of $\vdash$ are not translated in a straightforward fashion to $\leqq$. Every ambiguity logic satisfies (mon$\leqq$):

   (mon$\leqq$) $\alpha \leqq \alpha' \& \beta \leqq \beta'$ entail $\alpha \| \beta \leqq \alpha' \| \beta'$

   but this does *not* entail that $(\mathbf{I;}\ I)$ is sound. The underlying reason is that *prima facie* $\alpha \vdash \beta$ does neither imply that $\alpha \leqq \beta$ holds, nor vice versa.

3. The third difference is that we have different quantifier scope over contexts in case we have several premises, such as in $(\vee I)$. This rule states that if in a context on the left of $\vdash$, we can have both α and β, we can also have $\alpha \vee \beta$. But if we try to formulate a corresponding semi-inequality, the context does not occur. The following semi-equation is related to $(I\vee)$, but not equivalent.

   (41) $\alpha \leqq \gamma \& \beta \leqq \gamma \Rightarrow \alpha \vee \beta \leqq \gamma$

So we state the following:

Conjecture 3 *There is no set of equalities* $\boldsymbol{E}$ *such that* $\mathsf{cTAL} = \tau\mathsf{DAL}(\boldsymbol{E})$*, and no set of equalities* $\boldsymbol{E}$ *such that* $\mathsf{TAL} = \tau\bar{\mathsf{c}}\mathsf{DAL}(\boldsymbol{E})$.

6 Conclusion

6.1 Open Questions, Loose Ends

We start the conclusion with a list of prominent open problems and questions.

Commutative Logics of Trust So far, we have seen several commutative logics of trust. However, none of them was really convincing: cTAL turned out to be equivalent to $L_{\blacksquare\blacklozenge}$, a trivial logic. Logics like $\tau L_{\blacklozenge\blacksquare}$ and $\tau\mathsf{DAL}$, on the other hand, fail to satisfy fundamental properties: $\tau\mathsf{DAL}$ does not satisfy the laws for distributive lattices, and $\tau L_{\blacklozenge\blacksquare}$ does not even admit the fundamental ambiguation rule $(\mathbf{I;}\ I)$. There are two properties which we think a reasonable logic should have: firstly, it should be classically congruent (its inner logic extends classical logic), and secondly it should admit the rule $(\mathbf{I;}\ I)$. The question is: is there a reasonable commutative logic of trust? So far, we have not seen one, and we leave this question open here. Still, it is very striking that trust and commutativity do not go well together. We conjecture that trust requires keeping track of plausibility.

The issue of reducibility We have seen sequents of the form $p\|\neg p \vdash q\|\neg q$, which are valid even in a conservative logic as DAL, but also in TAL. Such a sequent can be called *irreducible*, since it is in ambiguous normal form, yet the unambiguous "submeanings" on the left/right are not in any classical entailment relation. Having a logic with irreducibles is paramount (modulo commutativity and associativity) to the rule (I**;** I) not being invertible. The only logic without irreducibles we have considered here is $L_{\blacklozenge\blacksquare}$, which however does not derive many reducible sequents either, like $p\|q \vdash p\|q$. So there should be an intermediate ambiguity logic which derives all and only reducible sequents. We plan to present results on (ir)reducibility in a separate publication.

Logics without e- and u-substitution? The Fundamental Theorem entails that there are four types of logics: 1. closed under u-substitution, not e-substitution (trustful), 2. closed under e-substitution, not u-substitution (distrustful), 3. closed under both (trivial), 4. closed under none of the two. It is this fourth type we have not observed yet. Given that we require closure under classical substitution, it does not seem to be straightforward to construct logics of this type. So do these logics exist, and are they interesting? We leave this open here.

Which logic is most adequate? In this article, we have made a plea for logical pluralism, or rather: we have shown that it is necessary. There simply is not one most plausible logic of ambiguity. But then, of course, some are more reasonable than others: for example, DAL seems to be a very reasonable logic of distrust, TAL a very reasonable logic of trust. Most of the other logics we presented have properties which are somehow strongly counterintuitive (even though also DAL and TAL have properties which are mildly counterintuitive). So the big conceptual (not mathematical) question is: which logic is adequate for which purpose? This is a question which can be approached from at least three different perspectives:

1. Cognitive science: how do humans actually reason? A very good case study (though in a slightly different setting) can be found in [6].

2. Computer science: [1] show that ambiguity almost inevitably enters into formal ontologies. Provided ontologies are based on fragments of first-order logic, which logic is the most apt for reasoners in ontologies? This is a complex question, which requires results on first-order extensions and decidable fragments for the propositional logics considered here.

3. Philosophy: It is also a philosophical question for (at least) argumentation theory. Of course this overlaps with 1., but this cannot be reduced to cognitive science (otherwise all logic were cognitive science).

We defer this to future work.

6.2 The Lesson to be Learned

What do we learn from this article? First and foremost, there *is* a field of ambiguity logics, with the Fundamental Theorem and the Trust Theorem as their main cornerstones. What they show is the following: there *cannot* be "the" logic of ambiguity: every non-trivial ambiguity logic necessarily lacks at least one fundamental closure property in the sense of [13], closure under substitution of equivalents or closure under uniform substitution. Moreover, this mathematical choice corresponds closely to a conceptual choice: whether we are in a situation of **trust** or **distrust**. Trust corresponds to closure under uniform substitution, distrust to its lack. We showed that every (non-trivial) distrustful logic can be extended to a (non-trivial) trustful logic, but *never* the other way around: hence trust (ceteris paribus) allows for more valid arguments than distrust, which confirms our intuition. It is this tight correlation between conceptual and mathematical properties which we consider most important and beautiful in this article.

In other cases, even if intuitions might be clear in the assumptions we make, they can leave us puzzled as we proceed to discover the consequences which follow from them. For example, the fact that there do not seem to be reasonable commutative logics of trust is surprising and somewhat puzzling. Also, the fact that every classically congruent (a reasonable criterion) logics of trust contains, as an *inner logic*, a distrustful logic, is a pleasant surprise. Even more surprisingly, the most prominent distrustful logic DAL is the inner logic of the commutative version of the most prominent trustful logic TAL, as well as the inner logic of the minimal ambiguity logic $L_{\blacklozenge\blacksquare}$.

Finally, the main lesson is that ambiguity is a field where we need logical pluralism, or, as [15] put it: logical consequence itself is ambiguous. This means that even beyond the most prominent logics $\bar{\mathsf{c}}\mathsf{DAL}, \mathsf{DAL}, \mathsf{TAL}, \mathsf{cTAL}$, there are interesting logics, which however have strange properties: they might not satisfy properties as the DeMorgan laws (as $\tau\mathsf{DAL}$), or not even satisfy $\alpha \vdash \alpha$ for all α (as $L_{\blacklozenge\blacksquare}$). We have to decide logical properties we consider desirable in which circumstances. And maybe the most interesting logics remain yet to be discovered!

References

[1] Alexandra Arapinis and Laure Vieu. A plea for complex categories in ontologies. *Applied Ontology*, 10:285–296, 12 2015.

[2] Philippe de Groote. Partially commutative linear logic: sequent calculus and phase semantics. In Michele Abrusci and Claudia Casadio, editors, *Third Roma Workshop: Proofs and Linguistic Categories*, pages 199–208, Bologna, 1996.

[3] Roy Dyckhoff, Mehrnoosh Sadrzadeh, and Julien Truffaut. Algebra, proof theory and applications for a logic of propositions, actions and adjoint modal operators. *Electr. Notes Theor. Comput. Sci.*, 286:157–172, 2012.

[4] Markus Egg. Semantic underspecification. *Language and Linguistics Compass*, 4(3):166–181, 2010.

[5] Tommaso Fornaciari, Alexandra Uma, Silviu Paun, Barbara Plank, Dirk Hovy, and Massimo Poesio. Beyond black & white: Leveraging annotator disagreement via soft-label multi-task learning. In *Proceedings of the 2021 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies*, pages 2591–2597, Online, 2021. Association for Computational Linguistics.

[6] Gregory Frost-Arnold and James Beebe. Confused terms in ordinary language. *Journal of Logic, Language and Information*, 29, 06 2020.

[7] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.

[8] Sara Negri and Jan von Plato. *Structural proof theory*. Cambridge Univ. Press, Cambridge [u.a.], 2001.

[9] Massimo Poesio. Semantic ambiguity and perceived ambiguity. In Kees van Deemter and Stanley Peters, editors, *Semantic Ambiguity and Underspecification*, pages 159–201. CSLI Publications, Stanford, CA, 1994.

[10] Greg Restall. *An Introduction to Substructural Logics*. Routledge, New York, 2008.

[11] Uwe Reyle. Dealing with ambiguities by underspecification: Construction, representation and deduction. *Journal of Semantics*, 10(2):123–179, 1993.

[12] Paul Saka. *How to Think About Meaning*. Springer, Dordrecht, 2007.

[13] Alfred Tarski. Über den Begriff der logischen Folgerung. *Actes du Congrés International de Philosophie Scientifique*, 7:1–11, 1936. translated as 'On the Concept of Logical Consequence' in [14], pp. 409–420.

[14] Alfred Tarski. *Logics, Semantics, Metamathematics*. Clarendon Press, Oxford, 1956.

[15] Kees van Deemter. Towards a logic of ambiguous expressions. In *Semantic Ambiguity and Underspecification*. CSLI, Stanford, 1996.

[16] Jan van Eijck and Jan Jaspars. Ambiguity and reasoning, 1995.

[17] Christian Wurm. The logic of ambiguity: The propositional case. In Annie Foret, Reinhard Muskens, and Sylvain Pogodalla, editors, *Formal Grammar - 22nd International Conference, FG 2017, Toulouse, France, July 22-23, 2017, Revised Selected Papers*, volume 10686 of *Lecture Notes in Computer Science*, pages 88–104. Springer, 2017.

[18] Christian Wurm. Reasoning with ambiguity. *Journal of Logic, Language and Information*, 2021.

[19] Christian Wurm and Timm Lichte. The proper treatment of linguistic ambiguity in ordinary algebra. In Annie Foret, Glyn Morrill, Reinhard Muskens, Rainer Osswald, and Sylvain Pogodalla, editors, *Formal Grammar - 20th and 21st International Conferences, FG 2015, Barcelona, Spain, August 2015, Revised Selected Papers. FG 2016, Bozen, Italy, August 2016, Proceedings*, volume 9804 of *Lecture Notes in Computer Science*, pages 306–322. Springer, 2016.

[20] David Yarowsky. Unsupervised word sense disambiguation rivaling supervised methods. In *Proceedings of the 33rd Annual Meeting on Association for Computational Linguistics*, ACL '95, page 189196, USA, 1995. Association for Computational Linguistics.