More than this There is another stronger logical property which corresponds to the trustful approach, namely **closure under substitution**, that, entailments are preserved by substituting atomic formulas uniformly by arbitrary formulas. In the distrustful approach, this *cannot* hold, since $\alpha \wedge \neg\alpha$ is a contradiction if and only if α is unambiguous; hence not every substitution of $p \wedge \neg p$ is a contradiction (same for tautologies). Now especially in connection with natural language, closure under substitution is pivotal, because the precise meaning of propositions is so hard to fix (e.g. who would claim that `dead` is ambiguous? Yet this is very well conceivable, considering example (14)). Lack of closure under substitution means being unable to say whether an argument is valid unless the precise structure of consituent propositions is fixed, and if the precise structure of propositions depends on natural language meaning, this is hardly ever going to happen. This is why we think the trustful approach is much more useful and relevant, and hence we will only pursue this approach.

3 Algebras of Ambiguity

3.1 Preliminaries and Boolean algebras

In this section, we will present an algebraic approach to the problem of reasoning with ambiguity. We will sketch the preliminaries, then present three relevant classes of algebras, prove the equivalence of their equational theories (i.e. the set of all equations holding in all algebras), which ultimately will lead us to discard this approach. The results of this section are thus mostly negative. If the reader is mainly interested in how ambiguity *can* be adequately treated, she can safely skip this section; the interesting result can be summarized as follows: algebra, at or least extensions of Boolean algebras, will not do the job. About the reasons for this we will speak in the end of this section; in section 6 we will see which general insights can be drawn from this.

The general setting we will use here are BOOLEAN ALGEBRAS, which are structures of the form $\mathbf{B} = (B, \wedge, \vee, \sim, 0, 1)$. As these are most well-known, we do not introduce them (the reader interested in background might consider (Kracht, 2003), (Maddux, 2006), or many other sources). We denote the class of Boolean algebras by $\mathbf{BA}$. In this section, we will only use elementary properties of Boolean algebras, frequently and without proof or explicit reference. Many results we present here depend on specific properties of Boolean algebras such as the law of double complementation; hence the results do depend on this very peculiar choice. However, there is a very good justification for this choice, namely that in semantics of natural language, which is by far the greatest field of research where ambiguity arises and has to be handled, there is (comparatively) very little work on approaches using non-classical logic.

In the algebraic approach to ambiguity, we think of the objects of algebras as propositional meanings; the operations of the algebra (in our case, the Boolean connectives and '$\|$') correspond to possibilities to combine these meanings. Here, the Boolean operations of course (loosely) correspond to their counterparts in natural language; for '$\|$', there is no corresponding connective. Importantly, there is no straightforward sense in which some meanings are more "basic" than others: all terms denote simple objects, that is, propositional meanings.

We now discuss what properties the connective $\|$ should satisfy on a conceptual level; put differently, we ask what kind of object is $a\|b$, and which rules does the operator $\|$ obey? We distinguish three different ways how we can conceptually conceive of the operation $\|$:

1. $a\|b$ denotes the "correct" meaning, that is, the one intended by the speaker (but which is unknown to any interpreter)
2. $a\|b$ is entailed by the "correct" meaning, that is, the one intended by the speaker
3. $a\|b$ is a "genuinely ambiguous" object, a sort of underspecification, which behaves in a certain combinatorial and inferential fashion

The most important property of $a\|b$ is that it includes an *epistemic* aspect in our algebra: because in cases 1. and 2., we refer to the intention of the speaker, which is invisible to any outsider. This is also clear in the case of 3., as in this case, we have a genuinely underspecified meaning, that is, one the true content of which we cannot reconstruct. In either case, these algebras seem to be somewhat more abstract than Boolean algebras, in that they lack a simple representation (as the one by powersets or topological spaces); however, we will provide some representation theorems below.

3.2 Three classes of algebras

We now introduce three classes of algebras corresponding to the three conceptions mentioned above. All of them will have the same signature $(A, \wedge, \vee, \sim, \|, 0, 1)$, where $(A, \wedge, \vee, \sim, 0, 1)$ is a Boolean algebra, and $\|$ is a binary operator. We adopt the following general conventions: boldface letters like $\mathbf{A}$ designate algebras, corresponding letters A denote their carrier set. We define $a \leq b$ as an abbreviation for $a \wedge b = a$ (equivalently, $a \vee b = b$). Another general convention we will adopt here is the following: let $\mathfrak{C}$ be a class of algebras, t, t' be terms over their signature. We write $\mathfrak{C} \models t = t'$ if for all $\mathbf{C} \in \mathfrak{C}$, all instantiations of t, t' with objects in C, the equality holds in $\mathbf{C}$. Hence we write $\mathbf{BA} \models a \vee \neg a = 1$ etc. The following algebras are ordered from strong to weak.

Strong ambiguous algebras In this class, we have the following axioms for $\|$:

$$(\|1) \qquad \sim(a\|b) = \sim a\|\sim b$$

$$(\|2) \qquad a \wedge (b\|c) = (a \wedge b)\|(a \wedge c)$$

$$(\|3) \qquad \text{At least one of} \quad a = a\|b \quad \text{or} \quad b = a\|b \quad \text{holds}$$

We denote the class of all algebras satisfying these axioms by **SAA**. ($\|1$) and ($\|2$) will hold in all classes, and it is these axioms which ensure universal distribution. ($\|3$) is the axiom peculiar to **SAA**, and all it states is that $a\|b$ either denotes a or it denotes b.

Weak ambiguous algebras In this class, we have the following axioms:

$$(\|1) \qquad \sim(a\|b) = \sim a\|\sim b$$

$$(\|2) \qquad a \wedge (b\|c) = (a \wedge b)\|(a \wedge c)$$

$$(\|3w) \qquad \text{At least one of} \quad a \leq a\|b \quad \text{or} \quad b \leq a\|b \quad \text{holds}$$

$$(\text{assoc}) \qquad a\|(b\|c) = (a\|b)\|c$$

We denote the class by **WAA**. As we see, it is only the slightly weaker equality in ($\|3w$) which distinguishes this from the strong form. Still, the two do not coincide. However, we will show that every weak ambiguous algebra is actually a universal distribution algebra. We need an additional axiom to ensure associativity, which is actually derivable in **SAA**, but does not seem to be derivable from the other axioms in **WAA**.

Universal distribution algebras

$$(\|1) \qquad \sim(a\|b) = \sim a\|\sim b$$

$$(\|2) \qquad a \wedge (b\|c) = (a \wedge b)\|(a \wedge c)$$

$$(\text{assoc}) \qquad a\|(b\|c) = (a\|b)\|c$$

$$(\text{inf}) \qquad a \wedge b \leq a\|b \leq a \vee b$$

$$(\text{mon}) \qquad a\|b \leq (a \vee c)\|(b \vee d)$$

This is the weakest algebraic class we present here, with actually "genuinely ambiguous" objects. As is easy to see, this class is a variety, being axiomatized by a set of (in)equalities. **SAA** and **WAA** are not varieties: every variety contains its free algebra generated by certain set; however, the free ambiguous algebra (both weak or strong), constructed as the one where all equalities hold which hold in all algebras, is not an ambiguous algebra, because of the disjunctive axiom. We will now consider the three classes one after the other.

3.3 Strong ambiguous algebras

We now present the most important results on the class of strong ambiguous algebras, which has been introduced and thoroughly investigated in (Wurm and Lichte, 2016).[5] Intuitively, this is a model where all ambiguous meanings exist, but every ambiguity is resolved to an underlying intention (this makes the implicit presupposition that ambiguous meanings are used consistently in one sense). This is a strong commitment, and mathematical results show that it is actually too strong. Firstly note, that ($\|1$),($\|2$) are sufficient for universal distribution: they entail all equations (9)–(11) (for details see Wurm and Lichte, 2016), as $\vee$ is redundant. The axiom (id) $a\|a = a$ is obviously derivable. Note, by the way, that ambiguous algebras have a very peculiar axiom, namely ($\|3$), which is a disjunction. This fact entails that if we construct the free ambiguous algebra over a given set of generators as the algebra with only the equalities which hold in all ambiguous algebras over this set – then this algebra is not an ambiguous algebra, since in general, $a \neq a\|b \neq b$. Hence **SAA** is not a variety. The concept will however not be entirely useless, as every free ambiguous algebra belongs to the class of universal distribution algebras. This nicely models the epistemic aspect of ambiguity: in the usage of ambiguous meanings, there always hold some equalities we do not know, and which do not hold in general (but rather by speaker intention). We now prove the main result on **SAA**, namely uniformity.

Lemma 1 *Let* **A** *be a strong ambiguous algebra, and* $a \in A$. *If* $a\|{\sim}a = a$, *then*

1. $\sim a\|a = \sim a$	*5.* $0\|\sim a = 0$	*9.* $\sim a\|0 = \sim a$
2. $1\|a = 1$	*6.* $a\|1 = a$	*10.* $0\|1 = 0$
3. $0\|a = 0$	*7.* $a\|0 = a$	*11.* $1\|0 = 1$
4. $1\|\sim a = 1$	*8.* $\sim a\|1 = \sim a$	

Proof. 1. follows by negation distribution; 2. is because $({\sim}a\|a) \vee a = 1\|a = {\sim}a \vee a = 1$. Results 3.–9. follow in a similar fashion from the distributive laws. To see why 10. holds, assume conversely that $0\|1 = 1$. Then we have

$$1 \wedge a = (0\|1) \wedge a = (0 \wedge a)\|(1 \wedge a) = 0\|a = a \tag{15}$$

– a contradiction to 3. 11. follows by distribution of $\sim$. □

Obviously, this lemma has a dual where $a\|{\sim}a = {\sim}a$, and where all results are parallel.

Lemma 2 *Let* **A** *be a strong ambiguous algebra. If for an arbitrary* $a \in A$, *we have* $a\|{\sim}a = a$, *then for all* $b, c \in A$ *we have* $b\|c = b$; *conversely, if we have* $a\|{\sim}a = {\sim}a$, *then for all* $b, c \in A$ *we have* $b\|c = c$.

Proof. We only prove the first part, the second one is dual.

Assume $a\|{\sim}a = a$, and assume $b\|c = c$. By the previous lemma, we know that $1\|0 = 1$, hence $(1\|0) \vee c = 1\|c = 1$, hence

[5] Actually, the matter is slightly more complicated: the authors defined **WAA** and "proved" it to be equivalent to **SAA**; this proof was however flawed.

$$(+)\ b = (1\|c) \wedge b = b\|(b \wedge c)$$

Conversely, we have $c = b\|c = (b\|c) \vee c = (b \vee c)\|c$. Now $b \wedge c = ((b \vee c)\|c) \wedge b = b\|(b \wedge c) = b$ by (+), hence

$$(\#)\ b \leq c$$

We also have $b = b \vee 0 = b \vee (0\|1) = b\|1$. By (#), $b = b \wedge c = (b\|1) \wedge c = (b \wedge c)\|(1 \wedge c) = b\|c$. Hence $b = c$ and the claim follows. □

Now we can prove the strongest result on **SAA**, the uniformity lemma.

Lemma 3 *(Uniformity lemma) Assume we have a strong ambiguous algebra* **A** *$a, b \in A$ such that $a \neq b$.*

1. *If $a\|b = a$, then for all $c, c' \in A$, we have $c\|c' = c$;*
2. *if $a\|b = b$, then for all $c, c' \in A$, we have $c\|c' = c'$.*

Proof. We only prove 1., as 2. is completely parallel. Assume there are $a, b \in A$, $a \neq b$ and $a\|b = a$. Assume furthermore there are $c, c' \in A$ such that $c\|c' \neq c$. There are two cases:

i) there is such a pair c, c' such that $c' = {\sim}c$. Then the dual result of the previous lemma leads us to a contradiction, because we then have $c\|{\sim}c = {\sim}c$, and consequently $a\|b = b$, which is wrong by assumption – contradiction.

ii) there are no such pair c, c'. Then however we necessarily have (among other) $a\|{\sim}a = a$, and by the previous lemma, this entails $c\|c' = c$ – contradiction. □

Put differently: let π_l be **left projection**, a binary function where $\pi_l(a, b) = a$; π_r is then **right projection**, with $\pi_r(a, b) = b$.

Lemma 4 *(Uniformity of ambiguous algebras) Every strong ambiguous algebra has the form $(B, \wedge, \vee, \sim, \pi_l, 0, 1)$ or $(B, \wedge, \vee, \sim, \pi_r, 0, 1)$, where $(B, \wedge, \vee, \sim, 0, 1)$ is a Boolean algebra.*

Hence for every Boolean algebra, there exist exactly two ambiguous algebras, one where $\|$ computes uniformly π_l for all arguments, and one where $\|$ computes uniformly π_r. This immediately entails another thing:

Corollary 5 *In all strong ambiguous algebras, if $a\|b = b\|a$, then $a = b$. Hence all commutative algebras contain one element only.*

We say an ambiguous algebra is right-sided or left-sided; we denote the left-sided algebra extending a Boolean algebra **B** by $C_l(\mathbf{B})$, the right-sided extension by $C_r(\mathbf{B})$. This entails that ambiguous algebras are rather uninteresting, as extending a Boolean algebra with a left/right projection operator is not very interesting. Hence even though the axiomatization seems unproblematic, ($\|$3) is too strong. We will therefore next consider algebras with weaker axioms for ambiguity.

3.4 Weak ambiguous algebras

It is obvious that **SAA** $\subseteq$ **WAA**, that is, every strong ambiguous algebra is a weak ambiguous algebra, since it easily follows from uniformity that strong ambiguous algebras satisfy the additional axiom (assoc) (simply by case distinction). What is less obvious is that **WAA** $\subseteq$ **UDA**; this is the first thing we will prove here. First we will prove that (id) holds in **WAA**.

Lemma 6 **WAA** $\models a = a\|a$.

Proof. $a \geq a\|a$: $a \wedge (a\|a) = (a \wedge a)\|(a \wedge a) = a\|a$.
$a \leq a\|a$: $a \vee (a\|a) = (a \vee a)\|(a \vee a) = a\|a$. □

Now this has an important corollary:

Corollary 7 **WAA** $\models a \wedge b \leq a\|b \leq a \vee b$.

Proof. We have $(a\|b)\wedge(a\wedge b) = (a\wedge(a\wedge b))\|(b\wedge(a\wedge b)) = (a\wedge b)\|(a\wedge b) = a\wedge b$ (by idempotence); hence $a \wedge b \leq a\|b$ by definition of $\leq$. Parallel for $\vee$: we have $(a\|b) \vee (a \vee b) = (a \vee b)\|(a \vee b) = a \vee b$. □

We now need two auxiliary properties which hold in all Boolean algebras:

B1 If $b \vee {\sim}a = 1$, then $a \leq b$.
B2 If $a \vee c = 1$ and $a \vee {\sim}c = 1$, then $a = 1$.

To see B1, consider that if $b\vee{\sim}a = 1$, then $a = (b\vee{\sim}a)\wedge a = (b\wedge a)\vee({\sim}a\wedge a) = (b \wedge a) \vee 0 = b \wedge a$, hence by definition of $\leq$, $a \leq b$. To see B2, consider that if the premise holds, we have $1 = (a \vee c) \wedge (a \vee {\sim}c) = a \vee (c \wedge {\sim}c) = a \vee 0 = a$. With this, we can prove that **WAA** satisfies (mon):

Lemma 8 *In* **WAA**, *the following equalities hold:*

1. $1 = 1\|({\sim}a \vee b \vee {\sim}c)\|1$
2. $1 = (1\|(a \vee c \vee {\sim}b)\|({\sim}a \vee b)\|1)$
3. $((a \vee c)\|b) \vee ({\sim}a\|{\sim}b) = 1$
4. $a\|b \leq (a \vee c)\|b$
5. $a\|b \leq a\|(b \vee c)$
6. $a\|b \leq (a \vee c)\|(b \vee d)$

Proof. 1. We have

$$\begin{aligned}1 &= (a\|b) \vee {\sim}(a\|b)\\ &= (1\|(a \vee {\sim}b)\|({\sim}a \vee b)\|1)\\ &= (1\|(a \vee {\sim}b)\|({\sim}a \vee b)\|1) \vee ({\sim}a \vee b \vee {\sim}c)\\ &= 1\|1\|({\sim}a \vee b \vee {\sim}c)\|1\\ &= 1\|({\sim}a \vee b \vee {\sim}c)\|1\end{aligned}$$

2. We use B2:

$$\begin{aligned}(1\|(a \vee c \vee {\sim}b)\|({\sim}a \vee b)\|1) \vee c &= (1\|(a \vee {\sim}b \vee c)\|({\sim}a \vee b \vee c)\|1)\\ &= (a\|b) \vee {\sim}(a\|b) \vee c\\ &= 1.\end{aligned}$$

and

$$\begin{aligned}(1\|(a\vee c\vee {\sim}b)\|({\sim}a\vee b)\|1)\vee{\sim}c &= 1\|1\|({\sim}a\vee b\vee{\sim}c)\|1\\ &= 1\|({\sim}a\vee b\vee{\sim}c)\|1.\end{aligned}$$

and the claim follows from 1.

3. is obtained from 2. by applying distributive laws.

4. Since ${\sim}a\|{\sim}b = {\sim}(a\|b)$, this is obtained from 3 and B1.

5. All steps to derive 4. can be made parallel to derive 5.

6. The relation $\leq$, defined by $\wedge$ or $\vee$, is obviously transitive, hence this follows from the previous. □

This is already sufficient for the following result:

Corollary 9 *Every weak ambiguous algebra is a universal distribution algebra.*

Are there weak ambiguous algebras which are not strong ambiguous algebras? This question can be answered positively: just take the algebra with the four elements $\{0, 1, 0\|1, 1\|0\}$, with the obvious order (the square). There is actually just one algebra with these elements, where necessarily we have $1\|0\|1 = 1$, $1\|0\|0 = 1\|0$ etc., that is, we have $a_1\|a_2\|a_3 = a_1\|a_3$. This can be easily proved to be a weak ambiguous algebra, but it is not a strong ambiguous algebra, as $0 \neq 0\|1 \neq 1$.

The next question is: are there universal distribution algebras which are not weak ambiguous algebras? Again, the answer is positive: take the UDA which extends the four element Boolean algebra over $\{0, a, b, 1\}$ by ambiguous objects, and take the object $a\|b$. Then $(a\|b) \vee a = a\|(a \vee b) = a\|1$; $(a\|b) \vee b = 1\|b$. If this were a weak ambiguous algebra, we would have either $a \leq a\|b$ or $b \leq a\|b$. If $a \leq a\|b$, then $a\|b = a \vee (a\|b) = a\|1$, and if $b \leq a\|b$, then $a\|b = (a\|b) \vee b = 1\|b$. However, every UDA can be completed to a strong ambiguous algebra in two ways (see lemma 20) without collapsing any elements of the underlying Boolean algebra; hence in the algebra over $\{0, a, b, 1\}$, we would either have $a = 1$ or $a = b$ or $b = 1$, which by assumption do all not hold. This proves the following:

Lemma 10 **SAA** $\subsetneq$ **WAA** $\subsetneq$ **UDA**.

This is already all we can say about this class: Being located between **SAA** and **UDA**, it does not seem to be particularly interesting.

3.5 Universal distribution algebras

In a sense, ($\|$3) and ($\|$3w) state that we use an ambiguous term with a given intention. This might be true if we think of sentences uttered by speakers. It is no longer true if we just think (for example) of the lexicon, where terms exist irregardless of any intention. **UDA** models ambiguity without the underlying intention. Regarding the axioms, (inf) regulates the relation $\leq$ between ambiguous and unambiguous objects; (mon) the relation $\leq$ between ambiguous objects. Both are necessary, otherwise we could construct algebras where the relations are not as desired. (mon) is of course a shorthand for:

$$(a\|b) \wedge ((a \vee c)\|(b \vee d)) = a\|b$$

Hence (mon) basically provides a particular kind of distribution. It is easy to see that (mon) amounts to a form of monotonicity: increasing the arguments of $\|$ increases the value of the function. The formulation we choose immediately entails that **UDA** is a variety, contrary to **SAA** and **WAA**. Note that in presence of distributive laws, (inf) is equivalent to (id): (id) entails (inf), because $(a\|b) \wedge (a \wedge b) = (a \wedge b)\|(a \wedge b) = a \wedge b$, and hence by definition of $\leq$, $a \wedge b \leq a\|b$; parallel for $a \vee b$. Conversely, (inf) entails idempotence, because then $a = a \wedge a \leq a\|a \leq a \vee a = a$. Hence **UDA** splits ($\|3$) into two weaker axioms. Note also that as (inf) is correct beyond doubt, the maybe more questionable (id) is inevitable. Note that (id) might look questionable as it entails already things like

$$\begin{aligned} a\|b &= a\|a\|b \\ &= a\|b\|b \\ &= (a\|b\|b) \vee (a\|a\|b) \\ &= a\|(a \vee b)\|b \end{aligned}$$

(We skipped some straightforward intermediate steps.) In **UDA**, also inequations such as the following "law of disambiguation" are satisfied:

$$\begin{aligned} (a\|b\|c) \wedge {\sim}b &= (a \wedge {\sim}b)\|(b \wedge {\sim}b)\|(c \wedge {\sim}b) \\ &\leq (a \wedge {\sim}b)\|0\|(c \wedge {\sim}b) \\ &\leq a\|c \end{aligned}$$

To get a better intuition on the structure of universal distribution algebras, we present some first results. We say a term t is in **ambiguous normal form**, $t = t_1\|...\|t_i$, and $t_1, ..., t_i$ are Boolean terms. The following is not difficult:

Lemma 11 *For every term t, there is a term t' in ambiguous normal form such that* **UDA** $\models t = t'$.

To see the first claim, just iterate the application of distributive laws. When we have a Boolean combination of ambiguous terms, the procedure of forming ambiguous normal forms leads to an exponential blow-up in the size of terms. This "problem" (if we want to consider it as such) will however turn out immaterial for **UDA**, once we have the margin lemma, which is the central result on **UDA**.

An interesting property is the following: let $t = t_1\|...\|t_i$ be a term in ambiguous normal form. One might conjecture that **UDA** $\models 1 = t$ iff **BA** $\models 1 = t_1, ..., 1 = t_i$. This is however not correct, as can be seen from the following:

$$\begin{aligned} 1 &= (a\|b) \vee {\sim}(a\|b) \\ &= (a \vee ({\sim}a\|{\sim}b))\|(b \vee ({\sim}a\|{\sim}b)) \\ &= (a \vee {\sim}a)\|(a \vee {\sim}b)\|(b \vee {\sim}a)\|(b \vee {\sim}b) \end{aligned}$$

Here, a and b are arbitrary. This can still be strengthened: first, as a special case, put $b \equiv {\sim}a$; then we have:

$$\begin{aligned} 1 &= (a \| {\sim}a) \vee {\sim}(a \| {\sim}a) \\ &= 1 \| (a \vee a) \| ({\sim}a \vee {\sim}a) \| 1 \\ &= 1 \| a \| {\sim}a \| 1 \end{aligned}$$

where a is arbitrary. So far, we have only used Boolean algebra axioms and universal distribution. With (id) and (assoc) we can derive:

$$\begin{aligned} 1 &= (1\|a)\|({\sim}a\|1) & \\ &= ((1\|a)\|(1\|a))\|({\sim}a\|1) & \text{(id)} \\ &= (1\|a)\|(1\|a\|{\sim}a\|1) & \text{(assoc)} \\ &= (1\|a)\|1 & \text{substitution of line 1} \end{aligned}$$

There is a parallel derivation (using $\wedge$ instead of $\vee$) for $0 = 0\|a\|0$, hence the following equalities are valid in **UDA**:

(16) $$0 = 0\|a\|0$$

(17) $$1 = 1\|a\|1$$

where a is arbitrary. From here we can prove the following:

Lemma 12 **UDA** $\models a = a\|b\|a$.

Proof. By cases:

Case 1 Assume $b \leq a$. Then $a = 1 \wedge a = (1\|b\|1) \wedge a = a\|(b \wedge a)\|a = a\|b\|a$.

Case 2 Assume $a \leq b$. Then $a = a \vee 0 = a \vee (0\|b\|0) = a\|(a \vee b)\|a = a\|b\|a$.

Case 3 Assume $a \not\leq b$, $b \not\leq a$. Then $(a\|b\|a) \wedge a = a\|(b \wedge a)\|a = a$ (by case 1), hence $a \leq a\|b\|a$; similarly, $(a\|b\|a) \vee a = a\|(b \vee a)\|a = a$ (by case 2), hence $a\|b\|a \leq a$, hence the claim follows. □

Hence in particular, $0\|1\|0 = 0$, $1\|0\|1 = 1$. Hence we have again a very strong result, definitely stronger than what our intuition tells us about ambiguity. In particular, this makes it problematic to include commutativity:

Lemma 13 *Let* $\boldsymbol{U}$ *be a universal distribution algebra, with* $a, b \in U$*. Then if* $a\|b = b\|a$*, we have* $b = a$*.*

Proof. Assume $a\|b = b\|a$. Then $b = b\|a\|b = b\|b\|a = b\|a = b\|a\|a = a\|b\|a = a$. □

Corollary 14 *Every commutative universal distribution algebra* **U** *has at most one element.*

Proof. For all $a \in U$, we then have $a\|1 = 1\|a$, hence $a = 1$; in particular, $0 = 1$. □

We can now show the following result, which characterizes **UDA** very neatly:

Lemma 15 *(Margin lemma) Let* $\boldsymbol{U}$ *be a universal distribution algebra. Then for all* $a, b, c \in U$*, we have* $a\|b\|c = a\|c$*; put differently,* $\mathbf{UDA} \models a\|b\|c = a\|c$.

Proof. (For simplicity, we now omit associative brackets)

$$\begin{aligned} a\|b\|c &= a\|c\|a\|b\|c && (a = a\|c\|a) \\ &= a\|c && (c = c\|a\|b\|c) \end{aligned}$$

□

So in the end, in UDA arbitrary ambiguities "boil down" to the margins of ambiguous terms: commutativity is excluded, and more than 2-fold ambiguity is meaningless in this class of algebras. Now this is obviously a problem, which basically excludes UDA as a realistic model for ambiguity. We will discuss a way out of this predicament in later; but before this, we will prove a useful representation theorem for **UDA**.

Definition 16 *We define the* ***canonical*** **UDA** *over two given Boolean algebra* $\boldsymbol{B}_1, \boldsymbol{B}_2$ *as the direct product algebra* $\boldsymbol{B}_1 \times \boldsymbol{B}_2$*, where all Boolean operations are defined pointwise, and the operation* $\|$ *is defined by* $(a, b)\|(c, d) = (a, d)$ *for all* $a, c \in B_1$*,* $b, d \in B_2$*.*

It is straightforward to check that this satisfies all **UDA**-axioms. Canonical **UDA** have a very simple structure, in that they only slightly extend product algebras. By the margin lemma, we will prove that every UDA has – up to isomorphism – this form. Given a $\mathbf{U} \in \mathbf{UDA}$, we define the relations θ_l, θ_r by

$$\begin{aligned} &a\theta_l b \text{ iff for all } c\text{, we have } a\|c = b\|c \\ &a\theta_r b \text{ iff for all } c\text{, we have } c\|a = c\|b \end{aligned}$$

These are equivalence relations for every carrier set U, and in fact they are congruences for all universal distribution algebras, that is:

Lemma 17 *If* $a\theta_l b$ *and* $c\theta_l d$*, then*

1. $\sim a\theta_l \sim b$,
2. $(a \wedge c)\theta_l(b \wedge d)$,
3. $(a \vee c)\theta_l(b \vee d)$, *and*
4. $(a\|c)\theta_l(b\|d)$

Same for θ_r*.*

Proof. 1. Assume $a\|x = b\|x$ for all x, hence also $a\|{\sim}x = b\|{\sim}x$. Then in particular, $({\sim}a)\|x = {\sim}(a\|({\sim}x)) = {\sim}(b\|({\sim}x)) = ({\sim}b)\|x$.

2. Assume $a\theta_l b$ and $c\theta_l d$; then $(a \wedge c)\|x = (a\|x) \wedge (c\|x)$ by the margin lemma; $a\|x = b\|x$, $c\|x = d\|x$, $(b\|x) \wedge (d\|x) = (b \wedge d)\|x$.

3. Parallel.

4. Assume $a\|x = b\|x$ for all x. Then $a\|c = b\|d$, hence for all x, $a\|c\|x = b\|d\|x$ (for each direction we only need one of the premises). □

We now define maps h_l, h_r with $h_l(x) = \{a : a\theta_l x\}$, $h_r(x) = \{a : a\theta_r x\}$ (that is, elements are mapped onto congruence classes). These are, by the famous results of general algebra, homomorphisms for arbitrary universal distribution algebras. Hence we can construct the two homomorphic images $h_l(\mathbf{U}) = (U_{\theta_l}, \wedge, \vee, \sim, 0, 1)$ (with the congruence classes as carrier set), and $h_r(\mathbf{U})$. We now define the map ϕ by $\phi(x) = (h_l(x), h_r(x))$. This is still a homomorphism for Boolean operations, because all operations are defined pointwise in the image algebra. The image $\phi[U]$ being a set of pairs, we can define $\|$ canonically by $(a,b)\|(c,d) = (a,d)$; as Boolean operations are defined pointwise, hence we obtain a universal distribution algebra which we denote by $\phi(\mathbf{U})$. The crucial lemma is the following (here $\cong$ denotes isomorphy of two algebras):

Lemma 18 $\phi(\boldsymbol{U}) \cong \boldsymbol{U}$.

Proof. We show two things, 1. $\phi(a\|b) = \phi(a)\|\phi(b)$ (that this holds for all other connectives already follows by general algebra), and 2. ϕ is a bijection.

1. Note that $\phi(a)\|\phi(b) = (h_l(a), h_r(a))\|(h_l(b), h_r(b)) = (h_l(a), h_r(b))$. By the margin lemma, we have $h_l(a\|b) = h_l(a)$, $h_r(a\|b) = h_r(b)$; hence $\phi(a\|b) = (h_l(a\|b), h_r(a\|b)) = (h_l(a), h_r(b))$.

2. ϕ is surjective by definition. Now assume we have $\phi(a) = \phi(b)$, hence $h_l(a) = h_l(b)$, $h_r(a) = h_r(b)$. Consequently, we have $a\|0 = b\|0$, and hence $a\|0\|a = b\|0\|b$. As $a = a\|0\|a$, $b = b\|0\|b$, it follows that $a = b$. □

Now as $\phi(\mathbf{U})$ is a canonical algebra for every $\mathbf{U}$, this proves the following theorem:

Theorem 19 *(Product representation theorem for* **UDA***) Every UDA is isomorphic a canonical UDA.*

This result shows that all **UDA** are very simple and well-behaved extensions of Boolean algebras. But it also shows, as did our results on **SAA**, that they are too simple to be really interesting!

3.6 Equivalence of equational theories

Before we present the central algebraic result of this article, we need the following lemma:

Lemma 20 *For every term t in the signature of* **UDA***, interpretation σ of t into a canonical universal distribution algebra* $\boldsymbol{U}$*, there exist two strong ambiguous algebras $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, with interpretations σ_1, σ_2 into A_1, A_2 such that $(\sigma_1(t), \sigma_2(t)) = \sigma(t)$.*

Proof. Assume we have the interpretation $\sigma : var \to B_1 \times B_2$. We know that for every $\mathbf{B} \in \mathbf{BA}$ there are exactly two strong ambiguous algebras with the same carrier set. We take the two algebras $(\mathbf{B}_1)_l$ and $(\mathbf{B}_2)_r$, that is their respective left and right completion; then we put $\sigma_1(x) = \pi_l(\sigma(x))$ and $\sigma_2(x) = \pi_r(\sigma(x))$.

We prove that these algebras and assignments do the job as required by an induction on the complexity of t. For atomic terms, the claim is straightforward, as $(\pi_l(\sigma(x)), \pi_2(\sigma(x))) = \sigma(x)$ by definition. Now assume the claim holds for some terms t, t'.

1. We have $\sigma_1(t \wedge t') = \sigma_1(t) \wedge \sigma_1(t') = \pi_l(\sigma(t)) \wedge \pi_l(\sigma(t')) = \pi_l(\sigma(t \wedge t')$ (by pointwise definition of $\wedge$); same for π_r and σ_2, hence $(\sigma_1(t \wedge t'), \sigma_2(t \wedge t')) = (\pi_l(\sigma(t \wedge t')), \pi_r(\sigma(t \wedge t'))) = \sigma(t \wedge t')$.

2. $\vee$ parallel.

3. $\sim$ similar.

4. $\|$ $\sigma_1(t\|t') = \pi_l(\sigma(t\|t')) = \pi_l(\sigma(t))$. Same for σ_2, so $(\sigma_1(t\|t'), \sigma_2(t\|t')) = (\pi_l(\sigma(t), \pi_r(\sigma(t'))$, which by canonicity entails the claim. □

Theorem 21 *For all terms t, t', the following three are equivalent:*

1. $\mathbf{UDA} \models t = t'$
2. $\mathbf{SAA} \models t = t'$
3. $\mathbf{WAA} \models t = t'$

Proof. 1. $\Rightarrow$ 2: $\mathbf{SAA} \subseteq \mathbf{UDA}$, hence the claim is obvious.

2.$\Rightarrow$ 1.: Contraposition: assume $\mathbf{UDA} \not\models t = t'$; hence there is $\mathbf{U}, \sigma$ for which the equality is false: $\sigma(t) \neq \sigma(t')$. Now we take an isomorphic canonical UDA, which we denote by $can(\mathbf{U})$, and which has the form $\mathbf{B} \times \mathbf{B}'$, where $\mathbf{B}, \mathbf{B}' \in \mathbf{BA}$. By the isomorphism ψ, we have $can(\mathbf{U}), \psi \circ \sigma \not\models t = t'$. Assume $\psi \circ \sigma(t) = (a, b) \neq (a', b') = \psi \circ \sigma(t')$. Now use the previous lemma: we have two strong ambiguous algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, interpretations σ_1 and σ_2, where $\sigma_1(t) = a$, $\sigma_2(t) = b$, $\sigma_1(t') = a'$, $\sigma_2(t') = b'$. Now as by assumption, either $a \neq a'$ or $b \neq b'$, we either have $\mathbf{A}_1, \sigma_1 \not\models t = t'$ or $\mathbf{A}_2, \sigma_2 \not\models t = t'$. Either way, $\mathbf{SAA} \not\models t = t'$, hence the claim follows.

1 $\Rightarrow$ 3 $\mathbf{WAA} \subseteq \mathbf{UDA}$, hence the claim is obvious.

3 $\Rightarrow$ 2 $\mathbf{SAA} \subseteq \mathbf{WAA}$, hence the claim is obvious. □

Hence we have three algebraic models, and all of them have the same equational theory, that is the same set of valid equations. This is, given the difference in axiomatization, rather astonishing and shows an interesting convergence. Unfortunately, we cannot consider this convergence as evidence for the "correct" model of ambiguity – because all algebras have strongly unintuitive properties. On the other hand, we do not see any algebraic alternatives either, because it seems impossible to weaken the axioms of **UDA** without losing essential properties of ambiguity. Before we sketch the way out of this dilemma, we will quickly present the (rather simple) corollaries on the decidability of the equational theories:

Corollary 22 *The equational theories of* **UDA**, **WAA**, **SAA** *are decidable; more precisely, their decision problem is NP-complete.*

Proof. We show the claim for **SAA**, from which all others follow. To check that $\mathbf{SAA} \models t = t'$, we just have to reduce the equation by interpreting $\|$ as π_l and π_r respectively, by which the equality reduces to two Boolean equalities $t_l = t'_l$, $t_r = t'_r$. Then the question is equivalent to checking whether $\mathbf{BA} \models t_l = t'_l$ and $\mathbf{BA} \models t_r = t'_r$, which is well-known to be NP-complete. $\square$

We will now quickly review one possible solution to the problem of the axioms being at the same time correct properties of ambiguity and "too strong". This solution is to use **partial algebras** and looks promising at first sight, but does not really lead out of our dilemmas.

3.7 Partiality

As we have mentioned above, a peculiar property of ambiguity in natural language is that it is – to our knowledge – never productive: ambiguities are in the lexicon and arise in syntactic derivations (and from some other sources), but we cannot construct them *ad libitum*, there is no productive mechanism for ambiguity. This nicely motivates the idea of algebras where $\|$ is a partial operation. Apart from this intuitive motivation of partiality, there is also a mathematical one: uniformity for **SAA** was derived from the existence of objects such as $a\|{\sim}a$, $0\|1$, which in natural language generally do not arise (leaving irony as part of pragmatics). The same holds for **UDA**, where proofs proceeded over peculiar objects like $0\|a\|0$ which need not necessarily exist. As **UDA** is the largest class of algebras we have presented and the only variety, we will present the results on partiality only for this class.

A **partial universal distribution algebra** is an algebra $(U, \wedge, \vee, \sim, \|, 0, 1)$, where $\|$ is a **partial** function $U \times U \to U$, which satisfies the usual equalities:

($\|$1) $${\sim}(a\|b) = {\sim}a\|{\sim}b$$

($\|$2) $$a \wedge (b\|c) = (a \wedge b)\|(a \wedge c)$$

(ass) $$a\|(b\|c) = (a\|b)\|c$$

(inf) $$a \wedge b \leq a\|b \leq a \vee b$$

(mon) $$a\|b \leq (a \vee c)\|(b \vee d)$$

Here equations have to be read in the following fashion: if one side of the equality is defined, so is the other, and obviously both are identical. Moreover, as the operations $\sim, \wedge, \vee$ are total, it follows that if $a\|b$ is defined, so are ${\sim}a\|{\sim}b$, $(a \wedge c)\|(b \wedge c)$ for all defined c, etc. Moreover, if $a\|(b\|c)$ is defined, so is $b\|c$, because undefined terms are absorbing for all operations. We now show that this extension does not really help.

Assume have a partial UDA $\mathbf{U}$, where $a, b \in U$, and $a\|b \neq \perp$ (we use $\perp$ as an abbreviation for *undefined*, not to be confused with 0!). Then we have the defined terms $1\|a \vee {\sim}b\|{\sim}a \vee b\|1 = 1$, $0 = 0\|a \vee {\sim}b\|{\sim}a \vee b\|0$. Here, $0\|1$ need not be defined, neither ${\sim}a\|a$. Still, we can conclude a number of things