Ambiguity in Language and Logic

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June 16, 2023

Abstract

Abstract. This article gives an explanation of how recent results on ambiguity logics are relevant to the linguistic and philosophical theory of ambiguity. To this aim, some fundamental definitions and results are explained. We formulate and provide evidence for three main hypotheses: Firstly, ambiguity is not a vague notion. Secondly, in (explicit) reasoning with ambiguity, we always have to consider the parameter ±trust. Thirdly, ambiguous propositions exist, but they cannot have the same rights and properties as unambiguous propositions; rather they should be considered "second class".

1. Introduction

There is a series of articles on the mathematics of reasoning with ambiguity (Wurm, C., tted; Wurm, C, 2021; Wurm, C and Lichte, T., 2016). This article should explain and explicate the meaning of formal results for the linguistic and philosophical theory of ambiguity. We will use only the indispensable amount of formalism, we will not present any new mathematical results and hence no proofs will be stated. Instead, we will introduce and explain the main notions and the conceptual meaning of previous results. So this article should state what linguists and philosophers can learn from previous work on ambiguity logics. To make the presentation more readable and interesting, we will illustrate this along three main hypotheses, which we will state below. The discussion of the hypotheses corresponds to the Sections 2,3 and 4. We believe that these hypotheses contain the core properties of ambiguity, and we believe that our work gives conclusive evidence they are correct.

There is a clear intuition: linguistic ambiguity comes from many sources (lexicon, morphology, syntax,...), but is a single phenomenon with common properties. Linguists and even non-linguists have a clear intuition on what ambiguity is, kids learn this often even before school. But as with almost any intuition, it is not difficult to shake it with boundary cases. For example, `plant` is ambiguous, no doubt. More complicated is the case of a word like `day`: it has (at least) three meanings which are clearly distinct: a 24h span, the span from sunrise to sunset, and the span from 0:00 to 23:59. A maybe more critical example is a word like `water`: in one reading (everyday use) it does not comprise ice, in a more technical reading it (arguably) does. A `granola` can denote the food to which you have to add the milk, or the food together with the milk. And maybe even more critical is a case like `dinner`, which is famously polysemous: `dinner was delicious`, `dinner took three hours`. So it does have (at least) two distinct meanings: an event, and a physical object/artefact. But is this ambiguity? In cases like these, formalization helps: definitions, motivated by clear cases, can help make our judgment more precise, which can then be re-applied to borderline cases.

Hypothesis 1 Ambiguity is not a vague phenomenon; there are no borderline cases of ambiguity. This is because the main properties of ambiguity are not denotational, but inferential and combinatorial.

The impression that there are borderline cases of ambiguity stems from a confusion of the *definitional criteria* of ambiguity with certain *typical properties*. For example, there has been an old debate on the difference between sense generality and ambiguity, and how to tell the two apart (see Zwicky, A. and Sadock, J., 1973). We think many arguments here were slightly besides the point, because the main difference does not lie in their different denotation, but how they combine with other meanings.

Hypothesis 2 The notion of trust is crucial for handling ambiguity and understanding ambiguity. In particular there is not and there cannot be *the* logic of ambiguity: we always have to set the parameter ±trust.

Intuitively, when handling ambiguity and making inferences from ambiguous statements, it is important to distinguish two types of situations: A) the statements have been made by somebody we do not trust, who might use ambiguous terms in different senses to mislead us. Alternatively, the statements might come from different sources/contexts, with the same effect, but without intention. B) we trust that ambiguous statements are reasonably used by a single source (for examples see below).

So reasoning with ambiguity has always to take into account whether we are in a trustful or distrustful setting.[1] Whereas it is intuitively clear that trust is important for reasoning with ambiguity, its central importance comes from the Fundamental Theorem. The Trust Theorem then shows that formal notions and conceptual notions are parallel: in a trustful setting, we accept more valid inferences than in a distrustful setting.

Hypothesis 3 Ambiguous propositions exist, but they are "second class": they do not have a semantics in terms of truth/falsity, but together with other propositions they do. This means that in some sense we need to assume the existence of ambiguous propositions, otherwise we cannot meet the basic (and unarguable) tenets for reasoning with ambiguity. On the other hand, we *cannot* assume that ambiguous propositions have the same properties as unambiguous propositions.

So are there ambiguous meanings? The answer is yes and no: it is unavoidable to assume the existence of ambiguous meanings, but an ambiguous meaning can never be a "first class citizen", by which we mean: propositions which are uniquely characterized by the set of situations/models in which they are true.

2. What is Linguistic Ambiguity?

2.1 *Fundamental Properties of Ambiguity*

First off, we have to clarify the notions of LOCAL and GLOBAL AMBIGUITY. A word like like `can` is ambiguous between a noun and an auxiliary, but this ambiguity will probably not make any sentence ambiguous, because morpho-syntactic context disambiguates; the ambiguity remains *local*. If we cannot disambiguate based on morpho-syntax, the ambiguity is *global*. Only global

[1]Distrust is thus intended in a very large sense, it also covers cases where ambiguous information comes from separate sources.

ambiguity is relevant for semantics, and local ambiguity will not play any role in this article.

Throughout this article, we denote ambiguity with the symbol $\|$, which we consider a propositional connective. Now we start with the FUNDAMENTAL PROPERTIES of ambiguity, which can be thought of as definitional criteria: if these criteria are not satisfied, we are not talking about ambiguity. These criteria have a twofold function: for natural language, they determine the phenomena that count as ambiguous; for logics, they determine what counts as an ambiguity logic.

Fundamental: Universal Distribution All connectives/operations uniformly distribute over ambiguity without altering it. We call this universal distribution, and it is the single most basic and characteristic feature of ambiguity. At the same time, it is the most useful to distinguish it from related phenomena. For example, `plant` is ambiguous, `vehicle` has a general sense. Denotationally, `plant` might denote an entity making photosynthesis, or a factory, `vehicle` might denote a car, a bike, a truck etc.

(1) a. `There is no plant.`
    b. `There is no vehicle.`

a. it also ambiguous. It does *not* mean there was no entity making photosynthesis *and* no factory. But b. means exactly this: there is none of the things that `vehicle` could denote: negated disjunction becomes the conjunction of negations (DeMorgan law), but ambiguity is invariant. This invariance property holds in all modifications:

(2) a. `There was a big plant.`
    b. `There are no big plants or buildings.`

In all these cases, we remain ambiguous; hence for every meaning function f, we have $f(\alpha\|\beta) = f(\alpha)\|f(\beta)$. This property is also very important for the treatment of ambiguity: every kind of (global) ambiguity in a natural language sentence gives ultimately rise to an ambiguity between sentence readings, just by "distributing up" This does obviously not mean that there cannot be ambiguity below the sentence level; but this ambiguity is either "cancelled out", or it "distributes up". So for a logical treatment, it is completely sufficient to consider ambiguity as a relation between propositions (which sloppily correspond to sentences). With propositional reasoning, universal distribution means

$$\text{(UD)}\quad \begin{array}{lll} \neg(\alpha\|\beta) & \text{is equivalent to} & \neg\alpha\|\neg\beta \\ (\alpha\|\beta)\vee\gamma & \text{is equivalent to} & (\alpha\vee\gamma)\|(\beta\vee\gamma) \\ (\alpha\|\beta)\wedge\gamma & \text{is equivalent to} & (\alpha\wedge\gamma)\|(\beta\wedge\gamma) \end{array}$$

Fundamental: Unambiguous Entailment An ambiguity between m_1 and m_2 entails their disjunction and is entailed by the conjunction. Formally, we can write

(UE) $\alpha \wedge \beta$ entails $\alpha\|\beta$ entails $\alpha \vee \beta$

This is a necessary condition, not a sufficient one: obviously it is also satisfied by sense generality. To make this property more restrictive, one could say that $\alpha\|\beta$ is intermediate between the two, but it does not coincide with either of them. This can be nicely illustrated:

(3) a. `Hand me over the pastry and the money!`
b. `Hand me over the dough!`[2]
c. `Hand me over the pastry or the money!`

a. entails b. entails c., but obviously none of them coincide.

Fundamental: Associativity This means that readings in ambiguous expressions do not come in groupings. Formally,

($\|$assoc) $\alpha\|(\beta\|\gamma)$ is equivalent to $(\alpha\|\beta)\|\gamma$

I think there is little to object to this. To clarify the meaning: can you think of two expressions which are ambiguous between exactly the same readings, but they are not equivalent because these readings are in a different groupings? We think intuition would clearly say they are equivalent.

Fundamental: Idempotence An ambiguity between m_1 and m_1 is equivalent to m_1

($\|$id) $\alpha\|\alpha$ is equivalent to α

This goes without comment. Note that if $\alpha\vee\alpha$ is equivalent to α is equivalent to $\alpha \wedge \alpha$, then this follows from (UE).[3]

[2] `Dough` is ambiguous between pastry and money.

[3] But this need not hold in all ambiguity logics for technical reasons.

Fundamental: Conservative extension This is not an empirical criterion, but purely logical. We think that a logical calculus of ambiguity should be a conservative extension of the classical calculus,[4] meaning that for formulas/sequents not involving ambiguity, exactly the same consequences should be valid as before. Conversely, this means even if we include ambiguous propositions, unambiguous propositions should behave as they used to before – new entailments should only concern ambiguous propositions. More simply: the existence of ambiguous propositions does not affect what unambiguous propositions mean and how they behave.

2.2 *Facultative Properties*

Secondly, there are FACULTATIVE PROPERTIES of ambiguity. These have a different status than the first category, in that whether we assume these properties to hold or not does not seem to change our linguistic notion of ambiguity, but rather how we formalize reasoning with ambiguity. Hence these properties are important to distinguish various ambiguity logics, not to define the linguistic phenomenon.

Facultative: Uniform Usage The property of (UU) is the conceptual counterpart to our (mathematical) distinction between trustful logics (as cTAL below) and distrustful logics (as DAL below). We formulate it as follows:

(UU) (Globally) ambiguous terms must be used consistently in *only one* sense.

There are good reasons for assuming (UU): Imagine someone telling you something about `plants`, and you struggle to understand what he is trying to tell. Now, for your interpretation it makes a huge difference whether you can reasonably assume that the term `plant` in the entire discourse is used consistently in one sense, or not. In the former case, you can try to disambiguate the term globally and from there make as much sense as possible. In the latter case, for each utterance, you have to take both readings into account. The classic work by YAROWSKY, D. (1995) gives evidence for consistent usage in texts. Note that even if we know that usage is consistent, we do of course not know which reading is being intended.

[4] Unless of course we assume that non-classical logics are more adequate to our purposes, but we will not consider this possibility here.

On the other hand, there are good reasons for refuting (UU): a formula of the form $\alpha \vee \neg\alpha$, with ambiguous α, need not be necessarily true in a very simple and reasonable interpretation of ambiguity (see van Eijck, J. and Jaspars, J., 1995). Dually, a sentence like

(4) `He is dead and he is not dead.`

is not necessarily a contradiction, `dead` could be used in two different senses, say medical and spiritual. Apart from this, we *can* use the same word with different meanings in the same sentence, as in `I spring over a spring in spring.` Another case where (UU) is not satisfied is when statements come from different sources or contexts. I can, on one occasion, explain to my kids: `Ice is water (in frozen form)`, on another occasion: `Ice is not water (and I told you to bring me water)`. Still, this is not inconsistent (more on ice and water below).

We will see below that (UU) corresponds to **closure under uniform substitution** in logics: whatever ambiguous term we use (as a substitute of an atomic proposition), we use it in *one sense*. In the distrustful approach (without (UU)), classical theorems are no longer generally valid if constituted by ambiguous propositions, and classical inferences (like Modus Ponens) usually fail if applied to ambiguous propositions, whereas in trustful reasoning, they remain valid (see also Wurm, C, 2021, for further discussion).

Facultative: Monotonicity This is a weaker property than (UU), but we prefer to make it optional to include some extreme examples of ambiguity logics (see $L_{\blacklozenge\blacksquare}$ below). Monotonicity means that every utterance should entail itself, and this property should be preserved over *weakening* modifications: `bank` entails `bank or restaurant`, or dually `plants and animals` entails `plants`. Formally,

($\|$mon) If α entails α', β entails β', then $\alpha\|\beta$ entails $\alpha'\|\beta'$

To see why this might be arguable: in `plants and animals`, the context might be thought of as disambiguating, whereas in `plants` there is no context. Also, a very distrustful agent might say that $\alpha\|\beta$ does not entail $\alpha\|\beta$, since the entailment is not true on any reading of both. Importantly, ($\|$mon) does not require that ambiguous terms are used consistently in one sense, it is strictly weaker (DAL satisfies ($\|$mon), but not (UU)).

Facultative: Law of Disambiguation Disambiguation of ambiguous statements is a fundamental reasoning operation. We state the Law of Disambiguation as follows:

(LoD) $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$

(here we subsume the case where α or γ is empty). Hence ambiguous formulas can be disambiguated to less ambiguous/unambiguous formulas. We will see that this law is not entirely unproblematic: it follows from ($\|$mon) and (UD), but only if we satisfy Uniform Usage (UU), hence it is satisfied in most logics of trust. For distrustful logics, LoD normally only holds for classical (unambiguous) β:

(wLoD) $\alpha\|\beta\|\gamma$ and $\neg\beta$ entail $\alpha\|\gamma$, provided β is unambiguous

Hence we have a weak LoD (unambiguous β), and a general LoD (arbitrary β). We will see the (very distrustful) logic $L_{\blacklozenge\blacksquare}$, which does not satisfy ($\|$mon), and hence does not even satisfy (wLoD). We do not think though that an ambiguity logic can be plausible without satisfying (wLoD).

Facultative: Commutativity This property states that in ambiguous propositions, the order of readings does not play a role, hence

($\|$comm) $\alpha\|\beta$ is equivalent to $\beta\|\alpha$

This is an arguable property: on the one hand, the meanings of an utterance can often be ordered in primary, secondary etc. On the other hand, we often cannot tell a natural ordering, and even if it is existent, we might want to disregard it. This property will play a role in the Fundamental Theorem. We will here only present commutative logics, because they are simpler to define.

2.3 *Prejudices*

Thirdly, there is a list of prejudices about ambiguity. These prejudices are quite tenacious and continue to exert an influence on the field, so it is important to recognize them as such.

Prejudice: Meanings are unrelated There is the prejudice that in "real (lexical) ambiguity", the readings of a word are unrelated. According to the prejudice, two words are homophonic due some phonological/historical accidents (whatever that may mean). Hence ambiguity should not be systematic, and not observable across many languages. If we want to explain what ambiguity is, we typically take a word like `plant`, because it illustrates the case so well. From here, it is a short way to think that in "typical ambiguity", the readings are unrelated, and they happen to sound the same because of some "historical accident". I think though that this kind of ambiguity is rather an exception, and that in most cases of ambiguity, meanings are related (`water`, `day`). What makes `plant` typical is rather the fact that it allows to illustrate the case in point very clearly.

To see that the prejudice is wrong, consider that even the classical `Every boy loves a woman` does not satisfy these properties: readings are obviously related by entailment, and the ambiguity surely is not an accident, given it occurs almost identically in a considerable number of languages. Also, the ambiguity of `day` does not satisfy it: the meanings are obviously related, and the ambiguity seems to exist consistently across a wide range of languages. This is a prejudice we have to get rid of.

Prejudice: Ambiguous terms have non-convex meanings Many scholars claim that the denotation of linguistic units are convex (see GÄRDENFORS, P., 2004, and many more recent articles), and hence convexity is a fundamental property of "meanings". However, this is obviously wrong in the case of many ambiguous expressions. So the way out is: these expressions do not have *one* meaning, but several meanings, and the expressions are ambiguous. Note the circularity: a meaning is convex, a non-convex meaning is not a meaning but ambiguity.

This might be correct (and anyway, we do not discuss this issue here). The problem arises when we then start using non-convexity as a *criterion* for ambiguity. Convexity is a rather vague criterion on meaning, unless we are talking about numeric quantifiers (e.g. `at least three`). For example, consider the examples `water` and `dinner`: is the dinner-event and the dinner-food convex? Is there something in between which is not denoted by `dinner`? That does not seem to be a question which can be reasonably discussed. Maybe even more intricate the question for metaphors, or generic/universal/existential uses of a word, for example `birds` in `Birds` have beaks (universal) and Birds lay eggs (generic). We think that non-convexity is another *typical feature* of ambiguity. However, it is not a reasonable *criterion*.

Prejudice: Humans always disambiguate It is true that humans often effortlessly disambiguate utterances, up to the point of not even noticing ambiguity. But it is not true that they *always* do this, and it is neither true that disambiguations are always unique. Rather than that, many utterances remain ambiguous, but humans do not seem to have any trouble with understanding and reasoning nonetheless (see for example Fornaciari, T. *et al.*, 2021). An easy illustration is given by inter-annotator agreement for anaphora resolution or word sense disambiguation: it is rarely above 90%, whereas the annotated text does usually not pose any problem to understanding (see also van Deemter, K., 1996; van Eijck, J. and Jaspars, J., 1995, for excellent expositions). An ambiguous utterance as

(5) The first thing that strikes a stranger in New York is a big car

obviously entails that in New York there are (rather many) big cars. Hence one fundamental insight underlying this work (and the whole topic) is that utterances often remain ambiguous, but this does not pose any serious problems for understanding and reasoning. Humans can perfectly reason with ambiguous utterances (more examples below).

Prejudice: Ambiguity is syntactic in nature This comes in many flavors: for example, some people sustain there is no ambiguous word plant, but two words $\texttt{plant}_1$, $\texttt{plant}_2$, which happen to look/sound the same (see Saka, P., 2007, for arguments against this). But also Montague's approach to quantifier scope ambiguity, as in

(6) Every boy loves a movie

is an example: there are two unambiguous semantic representations, depending on which (semi-)syntactic rules we use to construct the meaning. The trick is: the translation from form to meaning *is not functional* (or deterministic), so ambiguity never enters into semantics. Apart from the fact that this comes with problems of its own, one main problem is: in this case we cannot reason with ambiguity. We have to choose a single reading of a sentence in a more or less arbitrary fashion, or based on some local information, and then proceed with this. Sound (and complete) reasoning with ambiguity however presupposes that we have all readings available in some semantic representation (see also Wurm, C, 2021; van Eijck, J. and Jaspars, J., 1995).

Prejudice: Ambiguity is disjunction "Outsiders" tend to treat ambiguity simply as the *disjunction of meanings* (see Saka, P., 2007, for an overview and counterarguments). However, it has been observed very early on that this is necessarily inadequate (see for example Poesio, M., 1994): if we take an ambiguity $\alpha\|\beta$, where $\alpha \models \beta$, then $\alpha \vee \beta$ is obviously equivalent to β, and there would not even exist an ambiguity in any reasonable sense. Now this is exactly what happens in quantifier scope ambiguity, as in (6). A number of other problems arise: disjunction satisfies DeMorgan laws, ambiguity satisfies Universal Distribution.

Prejudice: Ambiguity is underspecification A very common approach for representing ambiguity (as e.g. in the quantifier case) is to use a sort of meta-semantics, whose expressions *underspecify* logical representations (see Reyle, U., 1993; Egg, M., 2010, for overview). Famous cases in point would be Cooper storage and Hole Semantics. The problem is: the metalanguage does not really constitute a satisfying semantic representation, unless it is itself a logic with a well-defined inference and meaning. If ambiguity becomes "underspecification", we still have to define the valid inferences and denotational properties of the underspecification language, otherwise we do not really meet the basic tenets of a semantics. On the other hand, if we start investigating this, we are back to the endeavor we are discussing here: ambiguity logics!

2.4 *Reconsidering Borderline Cases*

As we have said, people tend to mix certain "typical aspects" of "typical ambiguity" with its necessary and defining features. Having established the fundamental properties, let us reconsider the famous boundary case `water` (see Sennet, A., 2006, on the topic; we do not go deeply into the argument, we just think it illustrates our case very well).

(7) a. `Antarctica is earth's biggest reserve of sweet water.`
b. `On their descent from Nanga Parbat in winter, the alpinists were soon left without water.`

Obviously, in a. we mean water in any form, and most likely frozen. In b., the alpinists most likely had plenty of water in frozen form, but none in liquid. Having said this, both of the following utterances are ambiguous:

(8) a. `There is plenty of water! (in Antarctica/in a thermos jug)`
b. `We not enough water left! (on Nanga Parbat/on earth)`

Hence it is a rather clear case that `water` satisfies universal distribution and hence is ambiguous; testing (UE) etc. is a simple exercise. Note that Sennet, A. (2006) argues that this claim is not correct. We give a slightly modified counterargument; consider the following:

(9) a. `Ice is water.`[5]
b. `Water is a liquid.`
c. `Ice is a liquid.`

Under the ambiguity of water, both a. and b. might be acceptable (though they are not necessarily; it depends on whether we trust, see below). Now c. seems to follow, which is obviously wrong. Does this lead to paradox? No, because as proved in Wurm, C (2021), in trustful reasoning with ambiguity, *transitivity of inference is not sound.* Hence if we accept a. and b., this means that we trust (in that our interlocutor is not making misleading statements). But then we know (mathematically) that transitivity cannot be generally accepted! Note that similar "paradoxes" can be constructed for almost every ambiguous term:

(10) a. `A day is 24h`
b. `A day on a planet is one complete rotation.`
c. `One complete rotation of Jupiter is 24h.`

All these cases are smoothly resolved by ambiguity logics, as we will see in

[5] We assume the meaning of `x is y` is $[\![x]\!] \subseteq [\![y]\!]$, hence logically an implication.

Section 3.2.

2.5 *Hypothesis 1 – Revisited*

Hence the decisive properties of ambiguity are combinatorial and inferential, not denotational. The main distinctive feature is universal distribution, a property which we think only occurs with ambiguity. Since UD is a combinatorial feature, we think there can hardly be any vagueness in this regard. Critical cases can only arise if tests are not applicable. This might happen if

- the meanings are so strongly related that it is hard to get intuitions
- the meanings are related in a way that it is hard to apply a test, since most contexts disambiguate

The latter point applies to cases like polysemy: if we state `There was a dinner`, it means there was both event and food; but one can argue that this is due to world knowledge. However this means that it is hard to test for universal distribution, as in almost all contexts it is clear to what we refer (event, food, or both). But note that even if something is hard to determine, this does not imply that it is vague (see the famous example of `bearfast`, which means faster than the fastest bear up to the year 2000). We think this gives convincing evidence that ambiguity is not a vague notion.

3. What is an Ambiguity Logic?

3.1 *Logics with $\|$ and the Fundamental Theorem*

Let us quickly state what an ambiguity logic is. We consider classical logical languages enriched with an explicit ambiguity connective $\|$. Hence an ambiguity logic is a classical (propositional/predicate) logic extended with additional connective $\|$, which satisfies all the fundamental properties of section 2.1: (UD),(UE),($\|$assoc),($\|$id) and conservative extension. For simplicity, we only consider propositional logic. We take the usual conventions: $Var = \{p_1, p_2, p_3, ...\}$ is the set of propositional variables,

$$Form(\mathsf{AL}) ::= p \in Var \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \neg\alpha \mid \alpha \to \beta \mid \alpha\|\beta$$

is the set of (possibly) ambiguous formulas,

$$\mathit{Form}(\mathsf{CL}) ::= p \in \mathit{Var} \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \neg\alpha \mid \alpha \to \beta$$

is the set of classical formulas. A logic is characterized by its formula language *Form* and its consequence relation

$$\vdash \subseteq \mathit{Form} \times \mathit{Form},$$

where $(\alpha, \beta) \in \vdash$ is written $\alpha \vdash \beta$ and means α entails β. We denote classical logic by $\mathsf{CL} = (\mathit{Form}(\mathsf{CL}), \vdash_{\mathsf{CL}})$. An ambiguity logic L has the form $(\mathit{Form}(\mathsf{AL}), \vdash_L)$. We also use the notation $\alpha[\beta]$, which means that β is a subformula (one occurrence) in $\alpha[\beta]$. Importantly, this notation presupposes that $\alpha[-]$ does not contain negation or implication! We now present the Fundamental Theorem, which (we believe) is the most important result on ambiguity logics.

Theorem 1 *(Fundamental Theorem)*

1. *Let $\mathcal{L} = (\mathit{Form}(\mathsf{AL}), \vdash)$ be a logic which conservatively extends classical logic and satisfies (UD),($\|$assoc),(UE),($\|$comm), is closed under uniform substitution of atoms and admits the rule (cut). Then $\mathcal{L}$ is inconsistent.*

2. *Let $\mathcal{L} = (\mathit{Form}(\mathsf{AL}), \vdash)$ be a logic which conservatively extends classical logic and satisfies (UD),($\|$assoc),(UE), is closed under uniform substitution of atoms and admits the rule (cut). Then for all $\alpha, \beta, \gamma \in \mathit{Form}(\mathsf{AL})$, we have $\alpha\|\gamma\|\beta \dashv\vdash \alpha\|\beta$.*

The immediate consequence of the Fundamental Theorem is that every non-trivial ambiguity logic lacks at least one basic closure property of abstract logics in the sense of Tarski, A. (1936): either closure under uniform substitution, or closure under substitution of equivalents (or more strictly, transitivity). The proof of the Fundamental Theorem can be found in Wurm, C (2021), following up to Wurm, C and Lichte, T. (2016); Wurm, C. (2017). Hence the only way to incorporate the basic features of ambiguity into a non-trivial formal logic for reasoning with ambiguity is to abandon one fundamental feature of “normal logics” itself.

1. Uniform substitution of atomic propositions by arbitrary formulas preserves the truth of sequents: $\alpha \vdash_L \beta$, $\sigma : \mathit{Var} \to \mathit{Form}(\mathsf{AL})$ a uniform substitution, entail $\sigma(\alpha) \vdash_L \sigma(\beta)$. We call this **closure under u-substitution**.

2. Substitution of arbitrary $\dashv\vdash_L$ equivalent formulas preserves the truth of sequents: $\alpha \dashv\vdash_L \beta$ and $\gamma[\alpha] \vdash_L \delta$ entail $\gamma[\beta] \vdash_L \delta$; same on the right. We call this **closure under e-substitution**.

Let us illustrate this with two examples.

Example: e-congruence in arithmetic Here is an obvious example for arithmetic:

$$\begin{aligned} (4+5)\cdot 3 &= 27 \\ 2+3 &= 5 \\ \therefore (4+(2+3))\cdot 3 &= 27 \end{aligned}$$

Substitution of equivalent terms preserve truth of equations in arithmetics.

Example: u-congruence in arithmetic A simple law which is valid in arithmetic is the distributive law. This validity is preserved over uniform substitution:

$$\begin{aligned} (x+y)\cdot z &= x\cdot z + y\cdot z \\ x &\mapsto z+x \\ \therefore ((z+x)+y)\cdot z &= (z+x)\cdot z + y\cdot z \end{aligned}$$

Hence one law really represents infinitely many laws, which are all subsumed as *instances* of this law.

These are two properties which we are used to consider as natural – but when we construct ambiguity logics, one of the two we have to let go! We will now provide two examples of ambiguity logics, where for each of them, one of the two properties is lacking. The simplest method to construct ambiguity logics is by means of the two truth operators $\blacksquare, \blacklozenge$, introduced by van Deemter, K. (1996). We define them as maps from ambiguous formulas to classical formulas:

$$\begin{aligned}
\blacksquare p &= p & \blacklozenge p &= p \\
\blacksquare(\alpha \wedge \beta) &= (\blacksquare\alpha) \wedge (\blacksquare\beta) & \blacklozenge(\alpha \wedge \beta) &= (\blacklozenge\alpha) \wedge (\blacklozenge\beta) \\
\blacksquare(\alpha \vee \beta) &= (\blacksquare\alpha) \vee (\blacksquare\beta) & \blacklozenge(\alpha \vee \beta) &= (\blacklozenge\alpha) \vee (\blacklozenge\beta) \\
\blacksquare(\neg\alpha) &= \neg(\blacklozenge\alpha) & \blacklozenge(\neg\alpha) &= \neg(\blacksquare\alpha) \\
\blacksquare(\alpha \to \beta) &= (\blacklozenge\alpha) \to (\blacksquare\beta) & \blacklozenge(\alpha \to \beta) &= (\blacksquare\alpha) \to (\blacklozenge\beta) \\
\blacksquare(\alpha\|\beta) &= (\blacksquare\alpha) \wedge (\blacksquare\beta) & \blacklozenge(\alpha\|\beta) &= (\blacklozenge\alpha) \vee (\blacklozenge\beta)
\end{aligned}$$

Note that for all $\alpha \in Form(\mathsf{AL})$, $\blacksquare\alpha, \blacklozenge\alpha \in Form(\mathsf{CL})$.

DAL, a logic of distrust DAL was first introduced by van Eijck, J. and Jaspars, J. (1995). We give a different but equivalent definition here (see Wurm, C., tted, for proof).

Definition 2 *$\alpha \vdash_{\mathsf{DAL}} \beta$ if $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$ and $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$.*

DAL is a very simple and convenient distrustful logic. At the same time, together with its non-commutative counterpart (which we do not present here) it is one of the most reasonable distrustful ambiguity logics. Obviously $p\vee\neg p$ is a theorem in DAL (actually, it is a theorem in every ambiguity logic, by conservative extension). However, the uniform substitution $\sigma : p \mapsto p\|q$ does not preserve this: $(p\|q) \vee \neg(p\|q)$ is not a theorem in DAL.

(11) $\blacksquare((p\|q) \vee \neg(p\|q)) = (\blacksquare p\|q) \vee (\neg\blacklozenge p\|q) = (p \wedge q) \vee \neg(p \vee q)$

which is not a classical theorem. It is easy to see the trick: negation changes $\blacksquare$ to $\blacklozenge$. So DAL is not closed under u-substitution.

cTAL, a logic of trust Next consider the logic cTAL, presented and investigated in Wurm, C. (tted) (for its non-commutative version see Wurm, C, 2021). We give the following simple definition, which again is not the original definition but an equivalent one.

Definition 3 *$\alpha \vdash_{\mathsf{cTAL}} \beta$ if $\blacksquare\alpha \vdash_{\mathsf{CL}} \blacklozenge\beta$*

It is not difficult to show that this logic is closed under u-substutition; this follows from the fact on the right side, $\blacklozenge$ makes formulas "more true", and on the left side, $\blacksquare$ makes formulas "more false". cTAL is a logic of trust, but not a very reasonable one (we only present it because of its simplicity). Actually, one can think of it as a logic of ingenuousness: if we have an ambiguity $\alpha = a_1\|....\|a_i$, $\beta = b_1\|...\|b_j$, then we have an entailment $\alpha \vdash_{\mathsf{cTAL}} \beta$ if one reading a_n entails one reading b_m (sufficient criterion, not necessary). It is now easy to see the following:

(12) $$\blacklozenge\alpha \quad \dashv\vdash_{\mathsf{cTAL}} \quad \alpha$$

(13) $$\alpha \quad \dashv\vdash_{\mathsf{cTAL}} \quad \blacksquare\alpha$$

However, we obviously do not have $\blacklozenge\alpha \vdash_{\mathsf{cTAL}} \blacksquare\alpha$ – put $\alpha = p\|q$, we then would obtain $p \vee q \vdash p \wedge q$, which is not derivable in cTAL (because of conservative extension). So cTAL is not closed under e-substitution and not closed under transitivity.

3.2 *Trust and Distrust, Water and Ice*

Generally, all non-trivial ambiguity logics fall into two kinds: logics of trust and logics of distrust, which corresponds to the closure properties they have:

Definition 4 *Assume $L = (Form(\mathsf{AL}), \vdash_L)$ is a logic (of ambiguity).*

1. *We say L is a* ***trustful logic****, if for every uniform substitution σ : $Var \to Form(\mathsf{AL})$, $\gamma \vdash_L \delta$ entails $\sigma(\gamma) \vdash_L \sigma(\delta)$*

2. *We say L is a* ***distrustful logic****, if 1. $\gamma[\alpha] \vdash_L \delta$, $\alpha' \vdash_L \alpha$ entail $\gamma[\alpha'] \vdash_L \delta$, and 2. $\gamma \vdash_L \delta[\beta]$, $\beta \vdash_L \beta'$ entail $\gamma \vdash_L \delta[\beta']$.*

Hence cTAL is trustful, DAL is distrustful. Note that this definition allows for the possibility of a logic being both trustful and distrustful at the same time – but the resulting logic is necessarily trivial by the Fundamental Theorem!

Water and ice in cTAL Recall the argument: P1) Ice is water. P2) Water is a liquid. C) Ice is a liquid. C) is obviously wrong. The point is that we use water in two different senses in P1), P2). How is this excluded by our logic? Firstly, let us consider the trustful case, where the premises P1),P2) might be valid, but the conclusion C) should not be. To stick with propositional logic, we simplify the example. We translate P1),P2) as follows:

(14) $\vdash_{\mathsf{cTAL}} p \rightarrow (p\|q)$ (equivalently: $p \vdash_{\mathsf{cTAL}} p\|q$)

(15) $\vdash_{\mathsf{cTAL}} (p\|q) \rightarrow q$ (equivalently: $p\|q \vdash_{\mathsf{cTAL}} q$)

All four sequents can be easily verified, just make the translation into classical logic. However, the transitive inference is not valid:

(16) $p \not\vdash_{\mathsf{cTAL}} q$

Hence in the trustful approach, $\alpha \vdash \beta$, $\beta \vdash \gamma$ does *not* entail $\alpha \vdash \gamma$ – we do not allow for transitive inferences in general, hence we can accept P1) and P2) without accepting C). (Our formal treatment is of course simplified, but it gives a clear idea how it scales up) And this property is not arbitrary, but *necessary* from a mathematical point of view (Fundamental Theorem), as well as desirable from a conceptual point of view (recall what I tell my kids about water and ice).

Now comes an interesting subtlety: we have translated the three statements P1),P2),C) into three sequents, which we could (conceptually) call *judgments*. What happens if we translate them into a single sequent/judgment? Every trustful ambiguity logic L accepts all inferences of the form

(17) $\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash_L \alpha \rightarrow \gamma$

(trustfulness $\cong$ closure under u-substitution) for all α, β, γ. So we obtain

(18) $p \rightarrow (p\|q),\ (p\|q) \rightarrow q \vdash_{\mathsf{cTAL}} p \rightarrow q$

So under this reading, the paradox arises. But a second thought reveals that this is exactly what we predict: if someone in one judgment asserts P1),P2), he violates uniform usage and is misleading us. Hence if we use a trustful logic, we get a wrong result - namely C). So in this case it is the trustful logic which is inappropriate!

Conceptually, in deriving a single judgment, the constraints on uniform usage of terms are "the same", whereas in two judgments they might be different! So in a trustful ambiguity logic, the "domain of trust" is always the **single judgment** (see Kracht, M., 2011, on this notion). Putting different judgments together, we can never be sure whether we can trust, because even if usage is consistent in one judgment, it need not be consistent across different judgments.

Water and ice in DAL In DAL, the potential paradox does not arise: simply because the premises P1) and P2) are not accepted: if we mistrust, `ice is water` is not acceptable; in our simplified propositional form, we have $\not\vdash_{\mathsf{DAL}} p \to (p\|q)$ and $p \not\vdash_{\mathsf{DAL}} p\|q$: both do not hold (as is easy to check). Same for $\not\vdash_{\mathsf{DAL}} (p\|q) \to q$ and $p\|q \not\vdash_{\mathsf{DAL}} q$.

So, formal ambiguity logics show how apparent paradoxes can smoothly be resolved. Finally, let us consider the following. Intuitively our conception of trust tells us that in a trustful environment, we would accept more valid arguments than in a distrustful setting. A good example is the following:

(19) `Peter loves plants.`
`If someone loves plants, he loves nature.`
$\therefore$ `Peter loves nature`

This argument is acceptable, provided that we trust that `plant` is used consistently in one sense here. Hence trust allows arguments to be valid, which have to be refuted under mistrust. How does this go together with our conception of trust, which is based on closure properties? Here comes the Trust Theorem, which clarifies these concepts:

Theorem 5 *(Trust Theorem)*

1. *Every distrustful ambiguity logic L can be extended to a unique smallest trustful ambiguity logic τL, where $\alpha \vdash_L \beta$ entails $\alpha \vdash_{\tau L} \beta$. Moreover, if L is non-trivial, then τL is non-trivial.*

2. *No trustful ambiguity logic can be extended to a non-trivial distrustful ambiguity logic: if L_t is a trustful ambiguity logic, L_d is a distrustful ambiguity logic, and $\alpha \vdash_{L_t} \beta$ entails $\alpha \vdash_{L_d} \beta$, then L_d is trivial.*

Recall that trust/distrust is an important parameter of ambiguity logics, but ambiguity logics are not uniquely determined by it: there are many trustful and distrustful ambiguity logics. However, the Trust Theorem shows: in trustful reasoning, there can never be less valid arguments than in distrustful reasoning. But every (non-trivial) distrustful logic can be extended to a (unique smallest non-trivial) trustful logic. Hence our intuitive conception "This is an acceptable argument, but only if we assume that ambiguous terms are used consistently" are mirrored by mathematical properties – even though we have never explicitly required them to hold!

3.3 *Hypothesis 2 - Revisited*

Trustful reasoning with ambiguity means: our arguments are abstract schemes, we can "fill" them with whatever propositions we want. This has a lot of advantages: consider linguistic applications. The meaning of words/phrases/sentences has always and probably will always remain somewhat opaque, and some residual ambiguity can never be excluded (usually it can be taken for granted). In a trustful approach, this does not affect the validity of arguments at all. Our impression is that in human interaction (non-pathological), trustful reasoning prevails by large quantity.

In distrustful reasoning with ambiguity, inferences can be copy-pasted arbitrarily. This sort of reasoning comes in very handy if we have various statements from different sources, and we want to determine whether something follows from them: if statements/judgments come from different sources, trustfulness is ingenuousness, because how should we guarantee uniform usage, even if there is no ill will?

The main point is: we have the choice, but we have to choose. Any logical approach to ambiguity has to decide whether it wants to have closure under uniform substitution, or closure under substitution of equivalents. We cannot have both! Whenever we speak of reasoning with ambiguity, we have to speak of trust.

4. Are there Ambiguous Propositions?

4.1 *Clarifications*

The notion of PROPOSITION is a very fundamental one in philosophy, and there is an ongoing discussion about its exact meaning. We cannot go into this topic here, so we take the very simple stance that propositions are "the bearers of truth values" (see McGrath, M. and Frank, D., 2020), ignoring propositional attitudes and the like. Hence:

> A (FIRST CLASS) PROPOSITION is something which is either true or false in every model/situation, hence it can be always assigned a truth value in $\{0, 1\}$.

This is a very narrow reading of proposition, and this does obviously not hold for ambiguous propositions: an ambiguous proposition $p||q$ cannot be assigned such a truth value in a model which verifies p, falsifies q, without running into paradoxical consequences. Hence we want to introduce the larger notion of a "second class proposition". Of course we could discuss the details and technicalities of this notion endlessly, and the following is only one "working definition" among many.

> A formula α represents a SECOND CLASS PROPOSITION, if there is a (first class) proposition β such that $\alpha \wedge \beta$ is a (first class) proposition which is distinct from β.

For example, $p\|q$ is under most reasonable ambiguity logics not a first class proposition, as for example in DAL: we cannot assign it a binary truth value in $M = \{p\}$. But in the same logic $(p\|q) \wedge \neg q$ *is* a proposition: $M \models (p\|q) \wedge \neg q$ entails that $M \models p$, since DAL satisfies (wLoD), and so $M \models (p\|q) \wedge \neg q$ entails $M \models p$. And since '$\wedge$' is interpreted classically, $M \models (p\|q) \wedge \neg q$ entails $M \models \neg q$. Hence for a semantics of DAL (see VAN EIJCK, J. and JASPARS, J., 1995; WURM, C., tted, for details), we necessarily have

$$M \models (p\|q) \wedge \neg q \text{ iff } M \models p,\ M \not\models q.$$

Hence truth in a model is uniquely determined.

In every reasonable ambiguity logic, ambiguous formulas represent second class propositions: by UD, every ambiguous formula is equivalent to a formula $\alpha_1\|...\|\alpha_i$, where $\alpha_1, ..., \alpha_i$ are classical. Then $(\alpha_1\|...\|\alpha_i) \wedge (\neg\alpha_2 \wedge ... \wedge \neg\alpha_i)$ represents a proposition (we have to consider some special cases though, where "submeanings" are contradictory; we have to slightly generalize the argument in the above example).

4.2 *Ambiguous Propositions Cannot be First Class*

The fact that ambiguous propositions cannot be first class already follows from the Fundamental Theorem, but we will try to make the point clearer. Assume ambiguous propositions are first class, hence in every model they have a truth value in $\{0, 1\}$. The inference relation $\vdash$ is equivalent to $\leq$ in terms of truth values in every model. This means we have transitivity/e-congruence, because $\leq$ is transitive and we can obviously exchange all formulas with identical truth values. On the other hand, we also have closure under u-substitution: an inference $\alpha \vdash \beta$ holds for arbitrary assignments of truth values for propositional variables, so if we substitute variables with arbitrary formulas – which by assumption also have truth values – the inference remains valid. By the Fundamental Theorem, this leads to triviality.

Of course this argument only works in the context of classical logic. If we assume for example that our base logic is Kleene three-valued logic, and that both ambiguous propositions and atomic propositions can be assigned the value $\frac{1}{2}$, triviality does not follow. But in this case we abandon the premise that we conservatively extend classical logic.

4.3 *There have to be Ambiguous Propositions*

Let us assume for sake of argument that there are no ambiguous propositions. What would that mean? At first glimpse, a reasonable (and maybe commonly found) position would be the following: speakers do not process ambiguous information in a semantic sense. Ambiguous *utterances* (i.e. linguistic entities) are remembered literally, and we only operate on the readings, which we retract "on demand".

Having this position, we guess that one basic operation would be **disambiguation**. If we represent our (unambiguous) knowledge in a theory T, we can check whether reading r_i is consistent with T, and if not, discard it. But that already leads to trouble: we have an utterance u (which is not a semantic object), from which we can retrieve a list of readings $\{r_1, ..., r_n\}$. But: once we have disambiguated by excluding reading r_i, what is the representation for this? u itself is now useless (we would lose the disambiguation). But there is no other representation in our logic (by assumption), unless $n = 2$, where only one reading remains! As a next problem, consider the following inference:

(20) $p\|q,\ (\neg q)\|r,\ (\neg r)\|p$ entail p

This inference is intuitively sound (even under distrust), since under the assumption $\neg p$ we have q (by wLoD), hence r (by double negation congruence and wLoD), hence p, hence inconsistency. But how are we supposed to handle this without having ambiguous propositions? For a linguistic example, consider the following:

(21) a. Every boy loves a movie.
b. Some boy hates every movie.

Both are dually ambiguous, but together they clearly entail the unambiguous weak readings of both a. and b.: there cannot be one movie which is loved by every boy, and there cannot be one boy hating all movies. But this cannot be inferred via disambiguation, because we only have two binary ambiguities! Hence the retraction-disambiguation model fails, even for simple examples.

A simple argument shows that we actually need ambiguity logics. Assume that we subscribe to our above fundamental properties of ambiguity. This means, that for our utterance u with readings $\{r_1, ..., r_n\}$, the following holds: if $r_1 \vee ... \vee r_n \vdash \beta$, then u entails β, and dually, if $\beta \vdash r_1 \wedge ... \wedge r_n$, then β entails u. Now, this is a necessary criterion for entailment with ambiguity, and being minimalist, we can assume it is sufficient, that is, we assume it is the only criterion for inference. Using our truth operators, this means:

(22) $\alpha \vdash_{\blacklozenge\blacksquare} \beta$ iff $\blacklozenge\alpha \vdash_{\mathsf{CL}} \blacksquare\beta$

(note the inverted order of $\blacksquare, \blacklozenge$ with respect to cTAL) But: this defines already an ambiguity logic, in which ambiguous propositions are second class. In fact, it is the minimal ambiguity logic $L_{\blacklozenge\blacksquare}$ (see Wurm, C., tted).

Note by the way that $L_{\blacklozenge\blacksquare}$ satisfies only the fundamental properties of ambiguity and $\|$-commutativity. Uniform usage, monotonicity and even the weak law of disambiguation do not hold in this logic, neither does the inference (20). This however does not mean that ambiguous formulas are not second class propositions; rather it means that a semantics based on truth in models is incomplete for this logic.

4.4 *Hypothesis 3 - Revisited*

As we have said, in every ambiguity logic, ambiguous formulas represent second class propositions, and the minimal reasonable inferences scheme for ambiguity already entails we have such a logic ($L_{\blacklozenge\blacksquare}$). We have also seen that by the Fundamental Theorem, assuming first class ambiguous propositions leads to triviality. However, we have to be careful:

(23) $(\alpha_1\|...\|\alpha_i) \wedge \neg\alpha_2 \wedge ... \wedge \neg\alpha_i$

without a doubt represents a proposition (equivalent to $\alpha_1 \wedge \neg\alpha_2 \wedge ... \wedge \neg\alpha_i$). However, this does *not* mean that in every ambiguity logic L we have

(24) $(\alpha_1\|...\|\alpha_i) \wedge \neg\alpha_2 \wedge ... \wedge \neg\alpha_i \equiv_L \alpha_1 \wedge \neg\alpha_2 \wedge ... \wedge \neg\alpha_i$

In fact, it is easy to check that this does not hold in neither of $L_{\blacklozenge\blacksquare}$, cTAL nor DAL. The reason is that PROPOSITION is a SEMANTIC NOTION based on truth, and for all but few ambiguity logics the semantics based on the truth scheme

$$\alpha \vdash \beta \quad \text{iff} \quad M \models \alpha \Longrightarrow M \models \beta$$

(for some kind of model and interpretation) is **incomplete**.[6] Actually the truth based semantics is incomplete for *all* trustful ambiguity logics (for reasons we cannot lay out here), and hence this cannot be a reasonable criterion to favor one logic over another.

Maybe we can phrase our result in a different terminology. It is plain obvious that propositions in the first class sense are semantic objects: they can be entirely characterized by the situations in which they are true. For propositions in the second class sense, this is less clear: for the linguist, they would probably qualify as being semantic, whereas for the logician, they are rather syntactic (being objects of proof theory more than model theory). We would state ambiguous propositions are somewhat intermediate: semantic from a linguistic perspective because we can make inferences with them, but still not fully semantic because we cannot assume they have binary truth values.

This reminds of the general observation: ambiguity is somehow intermediate between syntax and semantics: it is surely not a simple semantic phenomenon such as conjunction or disjunction; but it is also not a syntactic phenomenon, which is eliminated before we interpret. This is an old observation (see conclusion of WURM, C, 2021), hence maybe the case was clear from the beginning. Nonetheless we hope to have made some ideas more precise in this discussion.

5. The Lesson

We hope that reading this article gives anyone from any background a more precise idea on linguistic ambiguity and its semantic status. And we hope that also the non-logician will be convinced that recent work on ambiguity

[6] This means it does not match the consequence relation of the logic. DAL can be embedded in such a scheme, see WURM, C. (tted); this is because on the left/right of $\vdash_{\mathsf{DAL}}$ we use the *same* truth operator, as opposed to cTAL, $L_{\blacklozenge\blacksquare}$.

logic is highly relevant for understanding linguistic ambiguity. Just to mention a few examples: the apparent paradox of water and ice resolves smoothly once we know the Fundamental Theorem. The same result also shows that there cannot be ambiguous propositions as "first class citizens", a fact which is not surprising, but yet had to be formally proved. A maybe less intuitively clear result is that in fact there are ambiguous propositions in a weaker sense which we have called second class. What is arguable is the underlying definition of "second class", for which there are many choices; we think however that under any reasonable choice the result stays the same.

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