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## THE AXIOMS AND ALGEBRA OF AMBIGUITY


#### Abstract

This paper continues a study of event ambiguity as a primitive concept. Axioms are described for a comparative ambiguity relation on an arbitrary event set that are necessary and sufficient for a representation of the relation by a functional that is nonnegative, vanishes at the empty event, and satisfies complementary equality and submodularity. Uniqueness characteristics of representing functionals are discussed. The theory is extended to multifactor events, where marginal ambiguity and additive representations arise.


Keywords: ambiguity, submodularity, axioms.

## 1. INTRODUCTION

This paper continues an investigation of axioms and numerical representations of event ambiguity begun in Fishburn (1991). The earlier paper proposed eight axioms for a comparative ambiguity relation $\gtrsim$ on a set $\mathscr{A}$ of events under the interpretation for $A \gtrsim B$ that event $A$ is at least as ambiguous or vague as event $B$. It also identified necessary and sufficient axioms for each of three increasingly restrictive numerical representations of ( $\mathscr{A}, \gtrsim$ ) when $\mathscr{A}$ is a finite Boolean algebra. My purpose here is to examine generalizations and extensions of the most restrictive representation when $\mathscr{A}$ may be infinite.

Throughout, $\mathscr{A}$ is a set of subsets of a state space $S$ (Savage, 1954). We refer to subsets of $S$ as events. The empty event $\varnothing$ and universal event $S$ are assumed to be in $\mathscr{A}$. They provide a natural origin for an ambiguity function $\alpha: \mathscr{A} \rightarrow \mathbb{R}$ via $\alpha(\varnothing)=\alpha(S)=0$. The complement $S \backslash A$ of event $A$ is denoted by $A^{c}$. If $\mathscr{A}$ is closed under complementation and finite unions, it is a Boolean algebra. We do not generally assume that these closure properties hold for $\mathscr{A}$.

Our conception of event ambiguity has roots in Keynes (1921) and Knight (1921). It was touched on in Savage's (1954, pp. 57-60) dismissal of second order probabilities, and popularized by Ellsberg's (1961) contention that judgments of comparative probability are affected by ambiguity in ways that contradict additivity of subjective
probability. Later writers, including Einhorn and Hogarth (1986), Curley and Yates (1989), Hazen (1989), Hogarth (1989), Schmeidler (1989), and Heath and Tversky (1991), embroider Ellsberg's theme in settings for decision making and choice behavior.

Since ambiguity is primitive in the present treatment, it is not viewed as an adjunct of subjective probability. Roughly speaking, probability is concerned with likelihood, whereas ambiguity or vagueness (Wallsten, 1990) is the antithesis of the clarity or specificity of envisioned events. A rare event may have low ambiguity (all 20 flips of a coin come up heads) or high ambiguity (humans will be extinct by the year 2200 ). Similarly, a highly probable event can be very ambiguous or very unambiguous. In numerical representations, the principal difference between subjective probability and ambiguity involves complementary events. For probability, $p(A)+p\left(A^{c}\right)=1$; for ambiguity, $\alpha(A)=\alpha\left(A^{c}\right)$. The latter equality, which is expressed axiomatically as $A \gtrsim A^{\mathfrak{c}}$ whenever $A, A^{\mathfrak{c}} \in \mathscr{A}$, reflects the intuition that whatever underlies the ambiguity of an event also underlies the ambiguity of its complement. When closure under complementation holds for $\mathscr{A}$, it is appropriate to think of ambiguity as a property of pairs $\left\{A, A^{\mathrm{c}}\right\}$ of complementary events.

The basic numerical representation of $(\mathscr{l}, \geq)$ that we consider involves both $\gtrsim$ and its asymmetric part $>$ defined by

$$
A>B \quad \text { if } \quad A \gtrsim B \text { and not } \quad(B \gtrsim A) .
$$

We also define $\sim$ by

$$
A \sim B \text { if } A \gtrsim B \text { and } B \gtrsim A,
$$

and say that $A$ and $B$ are incomparable (in regard to ambiguity) if not $(A \gtrsim B)$ and not ( $B \gtrsim A$ ). The qualitative system ( $\mathscr{A}, \gtrsim$ ) is defined to be representable if there exists $\alpha: \mathscr{A} \rightarrow \mathbb{R}$ that has the following properties for all $A, B \in \mathscr{A}$ :

$$
\begin{aligned}
& A \gtrsim B \Rightarrow \alpha(A) \geqslant \alpha(B) \\
& A>B \Rightarrow \alpha(A)>\alpha(B)
\end{aligned}
$$

$$
\begin{aligned}
& \alpha(\varnothing)=0 \\
& \alpha(A) \geqslant 0 \\
& \alpha\left(A^{\mathrm{c}}\right)=\alpha(A) \text { if } A^{\mathrm{c}} \in \mathscr{A} \\
& \alpha(A \cap B)+\alpha(A \cup B) \leqslant \alpha(A)+\alpha(B) \text { if } \\
& \qquad A \cap B, A \cup B \in \mathscr{A}
\end{aligned}
$$

The penultimate property is complementary equality, and the final property is submodularity. The initial properties require $\alpha$ to preserve $\gtrsim$ and $>$ in the usual fashion, and the others say that $\alpha$ is a nonnegative function that vanishes at the empty event. The implication $\alpha(S)=\alpha(\varnothing)=0$ matches our intuition that the empty and universal events are completely unambiguous.

A rationale for imposing submodularity is given in Fishburn (1991). To see what it presumes, suppose first that $A$ and $B$ are disjoint. Then submodularity reduces to $\alpha(\varnothing)+\alpha(A \cup B) \leqslant \alpha(A)+\alpha(B)$ or, since $\alpha(\varnothing)=0$, to the subadditive form

$$
\alpha(A \cup B) \leqslant \alpha(A)+\alpha(B)
$$

This expresses the idea that the levels of vagueness associated separately with events $A$ and $B$ will not compound superadditively when the two are conjoined. Put differently, the union of $A$ and $B$ may reduce or cancel ambiguities associated with each separately (consider $B=$ $A^{c}$ ), and will not introduce new sources of ambiguity in the combination that outweigh such reductions.

When $A$ and $B$ are not disjoint, let $C=A \cap B, A^{\prime}=A \backslash B$ and $B^{\prime}=B \backslash A$, so $A^{\prime}, B^{\prime}$ and $C$ are mutually disjoint. Then submodularity can be expressed as

$$
\alpha(C)+\alpha\left(A^{\prime} \cup B^{\prime} \cup C\right) \leqslant \alpha\left(A^{\prime} \cup C\right)+\alpha\left(B^{\prime} \cup C\right),
$$

which is the natural extension of subadditivity obtained by adding the same disjoint event $C$ to each term. The rationale for this case is similar to that given previously with a fixed event $C$ rather than $\varnothing$ as a common background.

The preceding representation is more general than the submodularity model of Fishburn (1991) in three ways. First, it relaxes the structure for $\mathscr{A}$ by not imposing finiteness or the closure properties of Boolean algebras. Second, it does not presume transitivity although it forbids cycles such as $\{A \gtrsim B, B \succsim C, C>A\}$, and it does not assume that $\gtrsim$ is complete. While no generality is lost if we presume $A \gtrsim A$, the representation allows widespread incomparability between events in $\mathscr{A}$. Third, we have omitted two restrictions on $\alpha$ used in Fishburn (1991), namely

$$
\begin{aligned}
\alpha(A)=\alpha(B) \Rightarrow & (\alpha(A \backslash B)=\alpha(B \backslash A) \quad \text { or } \\
& \alpha(A \cap B)=\alpha(A \cup B)), \\
\alpha(A)>\alpha(B) \Rightarrow & (\alpha(A \backslash B)>\alpha(B \backslash A) \quad \text { or } \\
& \alpha(A \cap B)>\alpha(A \cup B)) .
\end{aligned}
$$

These have intuitively defensible axioms such as $A>B \Rightarrow(A \backslash B>B \backslash A$ or $A \cap B>A \cup B$ ), but seem less central to ambiguity than the earlier properties. Since they have little effect on what follows and would complicate the representation, they will be suppressed.

The next section gives axioms for ( $\mathscr{A}, \gtrsim$ ) that are necessary and sufficient for $(\mathscr{A}, \gtrsim)$ to be representable. We use a finite cancellation condition for the finite $\mathscr{A}$ case, and an Archimedean axiom for the general case.

Section 3 discusses the family $F$ of all $\alpha$ that represent a particular $(\mathscr{A}, \gtrsim)$. We note that some $F$ contain both bounded and unbounded $\alpha$ (Example 1). When $\alpha\left(A_{0}\right)$ is fixed at 1 for a specific event $A_{0}$, the lower bound on $F$ thus restricted may (Example 2) or may not (Example 3) be a member of $F$.

Section 4 considers multifactor ambiguity with $S=S_{1} \times \cdots \times S_{n}$. When $\alpha$ is an ambiguity functional on the power set of $S$, its marginals on the $n$ factors, defined in a natural way, are also ambiguity functionals. Moreover, when $\alpha$ is additive on events $A_{1} \times A_{2} \times \cdots \times A_{n}$, $A_{i} \subseteq S_{i}$, as

$$
\alpha\left(A_{1} \times \cdots \times A_{n}\right)=\sum_{i=1}^{n} \beta_{i}\left(A_{i}\right), \quad \beta_{i} \geqslant 0
$$

the $\beta_{i}$ are the marginals of $\alpha$. The axioms of Section 2 are easily extended to necessary and sufficient conditions for representability with interfactor additivity. Finally, we sketch an axiomatization of the additive ambiguity model that extends well known axioms for additive conjoint measurement.

A brief discussion concludes the paper.

## 2. AXIOMS FOR REPRESENTABILITY

Because the models in Fishburn (1991) used $A \gtrsim B \Leftrightarrow \alpha(A) \geqslant \alpha(B)$, with $\mathscr{A}$ a finite Boolean algebra, their axioms were straightforward. Examples include weak order ( $Z$ is transitive and complete), $A \gtrsim \varnothing$, $A \sim A^{c},(A \sim \varnothing, A \cap B=\varnothing) \Rightarrow B \gtrsim A \cup B$, and a finite cancellation condition for submodularity.

The generality of our new representation for ( $\mathscr{A}, \gtrsim$ ) requires new axioms. We use two axioms which together are necessary and sufficient for representability. However, because of their generality, they are less intuitively straightforward than the former axioms.

The new axioms are a finite cancellation condition and an Archimedean axiom. The Archimedean axiom is needed only when $\mathscr{A}$ is infinite. Its omission yields a representation in an ordered field extension of $\mathbb{R}$ rather than in $\mathbb{R}$ when $\mathscr{A}$ is infinite which is similar to non-Archimedean or nonstandard representations in Narens (1985), Fishburn (1992a) and Fishburn and LaValle (1991).
To formulate the axioms, let $V$ be the vector space of all real valued functions $v$ on $\mathscr{A}$ for which $\{A \in \mathscr{A}: v(A) \neq 0\}$ is finite. Scalar multiplication and addition in $V$ are defined by $(\lambda v)(A)=\lambda v(A)$ and $(u+v)(A)=u(A)+v(A)$ when $\lambda \in \mathbb{R}$ and $u, v \in V$. In addition, $V_{1}+$ $V_{2}=\left\{v_{1}+v_{2}: v_{i} \in V_{i}\right\}$ when the $V_{i}$ are subsets of $V, \mathbf{0}$ denotes the function identically zero on $V$, and $\langle v, f\rangle$ is the inner product of $v \in V$ and $f: \mathscr{A} \rightarrow \mathbb{R}$.

The representation can be phrased entirely in terms of nonnegative inequalities $\langle c, \alpha\rangle \geqslant 0$ and positive inequalities $\langle c, \alpha\rangle>0$ with every coefficient function $c$ a member of $V$ with values in $\{1,0,-1\}$. For example, if $A, A^{\mathrm{c}} \in \mathscr{A}$ then $\alpha(A)=\alpha\left(A^{\mathrm{c}}\right)$, or $A \sim A^{\mathrm{c}}$, is equivalent to $\alpha(A)-\alpha\left(A^{\mathrm{c}}\right) \geqslant 0$ and $\alpha\left(A^{\mathrm{c}}\right)-\alpha(A) \geqslant 0$. And $A>B$ is equivalent to $\alpha(A)-\alpha(B)>0$.

Let $N$ consist of 0 plus all such $c$ for which we wish to have $\langle c, \alpha\rangle \geqslant 0$, and let $P$ be the set of $c$ s for which we want $\langle c, \alpha\rangle>0$. There is one $c \in P$ for each strictly related pair $A>B$. Its nonzero values are $c(A)=1$ and $c(B)=-1$. Nonzero values for the several versions of $c \in N \backslash\{0\}$ are as follows with $A, B \in \mathscr{A}$ :

1. $A \gtrsim B, A \neq B: c(A)=1, c(B)=-1$
2. $\alpha(A) \geqslant 0: c(A)=1$
3. $-\alpha(\varnothing) \geqslant 0: c(\varnothing)=-1$
4. $\alpha(A)-\alpha\left(A^{\mathrm{c}}\right) \gtrsim 0, A^{\mathrm{c}} \in \mathscr{A}: c(A)=1, c\left(A^{\mathrm{c}}\right)=-1$
5. submodularity, $A \cap B, A \cup B \in \mathscr{A}, A \backslash B \neq \varnothing, B \backslash A \neq \varnothing$ :

$$
c(A)=c(B)=1, c(A \cap B)=c(A \cup B)=-1
$$

Only the nontrivial instances of submodularity are noted. The instances with $A \subseteq B$ are automatic. Note also for $P$ that the definition of $>$ forbids $A=B$ when $A>B$.

Our finite cancellation condition is

AXIOM 1. For all $m \in\{1,2, \ldots\}$ and all $c_{j} \in N \cup P$, if $c_{1} \in P$ then $\sum_{j=1}^{m} c_{j} \neq \mathbf{0}$.

If $\alpha$ satisfies the representation and the hypotheses of Axiom 1 hold, then $\left\langle\Sigma c_{j}, \alpha\right\rangle=\Sigma\left\langle c_{j}, \alpha\right\rangle>0$, so $\Sigma c_{j} \neq 0$. Hence finite cancellation is necessary for representability.

THEOREM 1. Suppose $\mathscr{A}$ is finite. Then $(\mathscr{A}, \geq)$ is representable if and only if Axiom 1 holds.

The sufficiency of Axiom 1 follows from well known linear separation results for finite systems of linear inequalities, and can be seen as a corollary of Theorems 1 and 2 in Fishburn (1992b). To explain the algebraic structure for the sufficiency proof, let $U^{*}$ denote the convex cone generated by nonempty $U \subseteq V$ :

$$
\begin{aligned}
U^{*}=\{ & \sum_{i=1}^{n} \lambda_{i} v_{i}: n \in\{1,2, \ldots\}, \lambda_{i}>0 \quad \text { and } \\
& \left.v_{i} \in U \quad \text { for all } i\right\} .
\end{aligned}
$$

Suppose Axiom 1 holds. If $P$ is empty, $\alpha=\mathbf{0}$ satisfies the representation. Suppose $P$ is not empty. Then $-N^{*}$ and $P^{*}$ are separated by a hyperplane through 0 that includes $P^{*}$ in an open half space determined by the hyperplane. This presumes that $\mathscr{A}$ is finite. A positive normal $\alpha$ to the hyperplane then gives $\langle v, \alpha\rangle>0$ for $v \in P^{*}$ and $\langle-v$, $\alpha\rangle \leqslant 0$ for $v \in N^{*}$.

To define an appropriate structure for the Archimedean axiom, we refer to a subset $K$ of $V$ as a convex cone if $K \neq \varnothing,(u, v \in K$, $0<\lambda<1) \Rightarrow \lambda u+(1-\lambda) v \in K$, and $(v \in K, \lambda>0) \Rightarrow \lambda v \in K$. A convex cone $K$ is without origin if $0 \notin K$, and is Archimedean if $u$, $v \in K \Rightarrow(\lambda u-v \in K$ for some $\lambda>0)$.

AXIOM 2. If $P \neq \varnothing$ then there exists an Archimedean convex cone without origin in $V$ that includes $N^{*}+P^{*}$.

THEOREM 2. ( $\mathscr{A}, \gtrsim)$ is representable if and only if Axiom 2 holds.

Theorem 2 is a corollary of Theorem 1 in Fishburn (1992b), which is a linear separation theorem for possibly infinite systems of linear inequalities, each with a finite number of nonzero terms. Given Axiom 2 with $P$ not empty, the sufficiency proof shows that there is a linear functional $\alpha$ on $V$ for which $\langle v, \alpha\rangle>0$ on $P^{*}$ and $\langle-v, \alpha\rangle \leqslant 0$ on $N^{*}$. Axiom 2 ensures separation when $\mathscr{A}$ is infinite, granting Zorn's lemma or the Axiom of Choice (Kelley, 1955; Fishburn, 1970). Moreover, it implies finite cancellation, so that Axiom 1 need not be stated explicitly for Theorem 2. The necessity of Axiom 2 is seen by noting that if $\alpha: \mathscr{A} \rightarrow \mathbb{R}$ satisfies the representation and $P \neq \varnothing$, then $\{v \in V:\langle v, \alpha\rangle$ $>0\}$ is an Archimedean convex cone without origin that includes $N^{*}+P^{*}$.

The situation addressed by Theorem 2 undergoes modest changes if we assume more for $\gtrsim$ or $\mathscr{A}$. The addition of completeness ( $A$, $B \in \mathscr{A} \Rightarrow(A \gtrsim B$ or $B \geqq A))$ strengthens the comparability part of the representation to

$$
A \gtrsim B \Leftrightarrow \alpha(A) \geqslant \alpha(B),
$$

the typical form for a weak order. If $\mathscr{A}$ is a Boolean algebra, slight simplifications occur in $N$ 's description. If we assume that $\mathscr{A}$ is a

Boolean algebra and $\gtrsim$ on $\mathscr{A}$ is a weak order, we can use axioms such as $A \gtrsim \varnothing$ and $A \sim A^{\text {® }}$, and replace $N$ by its specialization for submodularity (see A8 in Fishburn, 1991).

It appears that there is no sufficient axiomatization for ambiguity that is comparable in elegance to axiomatizations for extensive measurement, additive conjoint measurement, subjective probability, belief functions and other topics discussed, for example, in Herstein and Milnor (1953), Savage (1954), Fishburn (1970, 1988), Krantz et al. (1971), Roberts (1979), Suppes et al. (1989), Wakker (1989), Wong et al. (1991), and Wong et al. (1992). Most of these topics encourage bisections, interfactor tradeoffs, additive equality comparisons and other features of measurement scales that have tight uniqueness properties. There is no obvious aspect to our conception of ambiguity that has comparable constructive power. It is true that submodularity constrains $\alpha$ in certain ways, but it lacks the force of additive equality comparisons found elsewhere. However, the belief function axiomization in Wong et al. (1991) suggests that a tightening of our notion of ambiguity which extends its submodularity property may have a simple axiomatization when $\mathscr{A}$ is the family of all subsets of a finite $S$.

## 3. UNIQUENESS CONSIDERATIONS

The preceding remarks suggest that ambiguity functions for our basic representation do not have a simple scale type. However, we can identify some of their uniqueness structure and illustrate transformation possibilities.

Given ( $\mathscr{A}, \gtrsim)$, let $W$ be the set of all real valued functions on $\mathscr{A}$, and let

$$
\begin{aligned}
F= & \{\alpha \in W: \alpha \text { satisfies the properties that define } \\
& \text { representability }\} .
\end{aligned}
$$

PROPOSITION 1. If $F$ and $P$ are nonempty then $F$ is a convex cone without origin in $W$.

Proof. Convexity and multiplication by a positive scalar preserve the linear inequalities that define representability.

We proceed under a bit more structure. Henceforth in this section assume that $\mathscr{A}$ is a Boolean algebra, $\gtrsim$ is a weak order, and $P \neq \varnothing$, i.e., $A>B$ for some $A, B \in \mathscr{A}$. Then the ambiguity scale type for $F \neq \varnothing$ lies somewhere between an ordinal scale with origin $(A Z$ $B \Leftrightarrow \alpha(A) \geqslant \alpha(B), \alpha(\varnothing)=0)$ and a ratio scale.

Proposition 1 excludes the claim that $F$ is Archimedean in $W$. Our first example shows that $F$ need not be Archimedean, which must be the case when it contains both bounded and unbounded functionals.

Example 1. Let $S=\{1,2, \ldots\}$ and let $\mathscr{A}$ consist of all finite subsets of $S$ and their complements. Define $\alpha$ on $\mathscr{A}$ by

$$
\alpha(A)=\min \left\{|A|,\left|A^{\mathrm{c}}\right|\right\}
$$

and take $A \gtrsim B$ if and only if $\alpha(A) \geqslant \alpha(B)$. We have $\alpha(\varnothing)=0$, $\alpha(A) \geqslant 0$ and $\alpha(A)=\alpha\left(A^{c}\right)$. To check submodularity, note first that if $A$ and $B$ are finite then $|A \cap B|+|A \cup B|=|A|+|B|$. If both $A$ and $B$ in $\mathscr{A}$ are infinite, the use of complementary equality gives the conclusion that $\alpha(A \cap B)+\alpha(A \cup B)=\alpha(A)+\alpha(B)$. That is,

$$
\begin{aligned}
\alpha(A \cap B)+\alpha(A \cup B) & =\alpha\left((A \cap B)^{\mathrm{c}}\right)+\alpha\left((A \cup B)^{\mathrm{c}}\right) \\
& =\left|A^{\mathrm{c}} \cup B^{\mathrm{c}}\right|+\left|A^{\mathrm{c}} \cap B^{\mathrm{c}}\right| \\
& =\left|A^{\mathrm{c}}\right|+\left|B^{\mathrm{c}}\right| \\
& =\alpha(A)+\alpha(B) .
\end{aligned}
$$

Suppose $A$ is finite and $B$ is infinite. Then $\alpha(A \cap B)+\alpha(A \cup B)=$ $|A \cap B|+\left|(A \cup B)^{\mathrm{c}}\right|=|A \cap B|+\left|A^{\mathrm{c}} \cap B^{\mathrm{c}}\right| \leqslant|A|+\left|B^{\mathrm{c}}\right|=\alpha(A)+$ $\alpha(B)$. Hence $\alpha$ is submodular, so $\alpha \in F$.
In contrast to unbounded $\alpha$, define bounded $\beta$ on $\mathscr{A}$ by

$$
\beta(A)=\alpha(A) /[\alpha(A)+1] .
$$

We claim that $\beta \in F$. Clearly, $A \gtrsim B \Leftrightarrow \beta(A) \geqslant \beta(B), \beta(\varnothing)=0$, $\beta(A) \geqslant 0$ and $\beta(A)=\beta\left(A^{c}\right)$. To check submodularity, assume with no loss of generality that $A \backslash B$ and $B \backslash A$ are nonempty. Suppose $A$ and $B$
are finite, and let $a=|A \backslash B|, b=|B \backslash A|$ and $c=|A \cap B|$. Submodularity then holds if and only if

$$
\frac{c}{c+1}+\frac{a+b+c}{a+b+c+1} \leqslant \frac{a+c}{a+c+1}+\frac{b+c}{b+c+1} .
$$

This reduces to $a+b+2 c \leqslant 2(a+b+2 c+1)$, so submodularity holds when $A$ and $B$ are finite. If $A$ and $B$ are infinite, complementation gives the same conclusion. Suppose finally that $A$ is finite and $B$ is infinite. Since $\alpha(A \cup B)=\left|(A \cup B)^{c}\right|=\left|A^{\mathfrak{c}} \cap B^{\mathrm{c}}\right|$, submodularity requires

$$
\frac{|A \cap B|}{|A \cap B|+1}+\frac{\left|A^{\mathrm{c}} \cap B^{\mathrm{c}}\right|}{\left|A^{\mathrm{c}} \cap B^{\mathrm{c}}\right|+1} \leqslant \frac{|A|}{|A|+1}+\frac{\left|B^{\mathrm{c}}\right|}{\left|B^{\mathrm{c}}\right|+1}
$$

which is true since $x /(x+1)$ increases in $x \geqslant 0$.
Hence $\beta \in F$, so $F$ contains bounded as well as unbounded functions.

The next example shows how submodularity can confine $F$ to an easily described family of concave functions.

Example 2. Let $S=[0,2]$ and let $\mathscr{A}$ consist of all Lebesgue measurable subsets of $S$. Define $\alpha$ by

$$
\begin{aligned}
& \alpha(A)=\min \left\{\mu(A), \mu\left(A^{c}\right)\right\}, \\
& \quad(\mu \text { denotes Lebesgue measure })
\end{aligned}
$$

and take $A \gtrsim B \Leftrightarrow \alpha(A) \geqslant \alpha(B)$. The ambiguity function is maximized at 1 when $\mu(A)=1$. We focus on the normalized family

$$
F_{1}=\{\beta \in F: \beta(A)=1 \quad \text { when } \mu(A)=1\}
$$

and write $\beta(\gamma)$ for $\beta(A)$ when $\mu(A)=\gamma$. Thus $\alpha(\gamma)=\min \{\gamma, 2-\gamma\}$. We say that $\beta$ is concave if

$$
0 \leqslant \gamma-\delta \leqslant \gamma+\delta \leqslant 1 \Rightarrow \beta(\gamma) \geqslant[\beta(\gamma-\delta)+\beta(\gamma+\delta)] / 2
$$

It is easily seen that $\beta \in F_{1}$ if and only if $\beta(0)=0, \beta(1)=1, \beta$ is
strictly increasing and concave on $[0,1]$, and $\beta(\gamma)=\beta(2-\gamma)$ when $1 \leqslant \gamma \leqslant 2$. The properties other than concavity characterize representability apart from submodularity. Submodularity implies

$$
\begin{aligned}
& 2 \beta(\gamma) \geqslant \beta\left(\gamma_{1}\right)+\beta\left(\gamma_{2}\right) \text { when } 0 \leqslant \gamma_{1}<\gamma<\gamma_{2} \leqslant 1, \\
& 2 \gamma=\gamma_{1}+\gamma_{2}
\end{aligned}
$$

by choosing $A$ and $B$ with $\mu(A)=\mu(B)=\gamma, \mu(A \cap B)=\gamma_{1}$ and $\mu(A \cup B)=\gamma_{2}$. Hence submodularity implies concavity in the present example. Conversely, if $\beta$ satisfies the other properties and submodularity fails, then $\beta$ will not be concave.

The functions in $F_{1}$ are continuous on ( 0,2 ) but can be discontinuous at the end points of $[0,2]$, so $F$ is not Archimedean. Also, since

$$
\left(0 \leqslant \gamma \leqslant 1, \beta \in F_{1}\right) \Rightarrow \beta(\gamma) \geqslant \gamma=\alpha(\gamma),
$$

$\alpha$ is, in a manner of speaking, the least concave member of $F_{1}$. That is,

$$
\alpha(A)=\min \left\{\beta(A): \beta \in F_{1}\right\} \quad \text { for all } \quad A \in \mathscr{A},
$$

so the lower bound of $F_{1}$ is in $F_{1}$.
Our final example shows that the lower bound of $F_{1}$ can be well outside of $F_{1}$.

Example 3. Let $\mathscr{A}$ be the power set of $\{1,2,3,4\}$. Omitting braces for subsets, let $F_{1}$ be the family of all $\beta: \mathscr{A} \rightarrow[0,1]$ for which

$$
\begin{array}{ll}
\beta=0 & \text { for } \\
\beta=a & \text { for } \\
\beta=3 & 4,1234 \\
\beta=b & \text { for } \\
\beta=c & \text { for } \\
\beta=1,24 \\
\beta=1 & \text { for } \\
13,23,124,234 \\
\end{array}
$$

with

$$
\begin{aligned}
& 0<a<b<c<1 \\
& 1 \leqslant a+c \leqslant 1+b \\
& b \leqslant 2 a .
\end{aligned}
$$

Assume that $\gtrsim$ is preserved by the $\beta$ values. Along with ordering, the inequality restrictions are precisely those needed for submodularity. In particular

$$
\begin{aligned}
& (A=123, B=134) \Rightarrow 1 \leqslant a+c \\
& (A=13, B=34) \Rightarrow a+c \leqslant 1+b \\
& (A=123, B=124) \Rightarrow b \leqslant 2 a .
\end{aligned}
$$

With max $\beta(A)$ fixed at 1 , let $\beta_{0}(A)=\inf \left\{\beta(A): \beta \in F_{1}\right\}$. Then

$$
\beta_{0}(1)=1 / 2, \quad \beta_{0}(3)=0 \quad \text { and } \quad \beta_{0}(13)=1
$$

so submodularity fails decisively for $\beta_{0}$.

## 4. MULTIFACTOR AMBIGUITY

We conclude our technical analysis of ambiguity with remarks on situations characterized by multiple sources of ambiguity. For simplicity, we assume that the state space $S$ is the Cartesian product $S_{1} \times \cdots$ $\times S_{n}$ of $n \geqslant 2$ nonempty factor sets, and that $\mathscr{A}$ is the family of all subsets of $S$. As done previously, a real valued function on a Boolean algebra will be called an ambiguity functional if it is nonnegative, vanishes at $\varnothing$, satisfies complementary equality, and is submodular.

When $A \in \mathscr{A}$ is the product of subsets of the factors, say $A=$ $A_{1} \times \cdots \times A_{n}$, we write $\alpha\left(A_{1}, \ldots, A_{n}\right)$ in place of $\alpha\left(A_{1} \times \cdots \times\right.$ $\left.A_{n}\right)$. Let $\mathscr{A}{ }_{i}^{0}$ denote the family of all nonempty subsets of $S_{i}$. The set of all $\left(A_{1}, \ldots, A_{n}\right)$ which correspond to events in $\mathscr{A}$ is $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$ in union with $\{(\varnothing, \ldots, \varnothing)\}$, where the $n$-tuple of $\varnothing$ s corresponds to the empty event. Note that if $A_{2} \neq \varnothing$ then ( $\varnothing, A_{2}, \ldots, A_{n}$ ) corresponds to nothing in $\mathscr{A}$.

Our remarks on ambiguity for multifactor situations that go beyond those of preceding sections focus on the notion of marginal ambiguity. Let $\mathscr{A}_{i}=\mathscr{A}_{i}^{0} \cup\{\varnothing\}$. With an obvious bow to marginal probability, we define $\alpha_{i}$ on $\mathscr{A}_{i}$ from an ambiguity functional $\alpha$ on $\mathscr{A}$ by

$$
\begin{aligned}
& \alpha_{i}\left(A_{i}\right)=\alpha\left(S_{1}, \ldots, S_{i-1}, A_{i}, S_{i+1}, \ldots, S_{n}\right) \text { for } A_{i} \in \mathscr{A}_{i}^{0} \\
& \alpha_{i}(\varnothing)=0 .
\end{aligned}
$$

The marginal functions inherit characteristics of their parent.

PROPOSITION 2. Suppose $\alpha$ on $\mathscr{A}$ is an ambiguity functional. Then $\alpha_{i}$ on $\mathscr{A}_{i}$ is an ambiguity functional for $i=1, \ldots, n$.

Proof. Given $\alpha$ of the hypothesis, $\alpha_{i}(\varnothing)=0, \alpha_{i}\left(A_{i}\right) \geqslant 0$ for all $A_{i} \in \mathscr{A}_{i}^{0}, \alpha_{i}\left(A_{i}\right)=\alpha_{i}\left(A_{i}^{c}\right)$ because $\alpha_{i}\left(S_{i}\right)=0$ and

$$
\left(S_{1} \times \cdots \times A_{i} \times \cdots \times S_{n}\right)^{\mathrm{c}}=S_{1} \times \cdots \times A_{i}^{\mathrm{c}} \times \cdots \times S_{n}
$$

when $A_{i} \notin\left\{\varnothing, S_{i}\right\}$, and $\alpha_{i}$ obviously inherits submodularity from $\alpha$.

Given $\geq$ on $\mathscr{A}$, it is natural to define $z_{i}$ on $\mathscr{A}_{i}^{0}$ by

$$
\begin{aligned}
& A_{i} \gtrsim_{i} B_{i} \quad \text { if } \quad\left(S_{1}, \ldots, A_{i}, \ldots, S_{n}\right) \\
& \quad \gtrsim\left(S_{1}, \ldots, B_{i}, \ldots, S_{n}\right) .
\end{aligned}
$$

If $\gtrsim$ is a weak order, so is $\gtrsim_{i}$. As in conjoint measurement and other multifactor topics, we say that the factors are independent if, for all $i$ and all $A_{i}, B_{i}, C_{i} \in \mathscr{A}_{i}^{0}$,

$$
\begin{aligned}
& \left(C_{1}, \ldots, C_{i-1}, A_{i}, C_{i+1}, \ldots, C_{n}\right) \\
& \quad \succsim\left(C_{1}, \ldots, C_{i-1}, B_{i}, C_{i+1}, \ldots, C_{n}\right) \Leftrightarrow A_{i} \gtrsim_{i} B_{i} .
\end{aligned}
$$

Higher order independence concepts are defined similarly when subsets of two or more factors act as a unit.

Independence among factors suggests that if several $A_{i}$ are highly ambiguous then $A=\left(A_{1}, \ldots, A_{n}\right)$ as a whole will be highly ambiguous, and if all $A_{i}$ are relatively unambiguous then $A$ will have low ambiguity. Indeed, our independence conditions are necessary for the additive form

$$
\alpha\left(A_{1}, \ldots, A_{n}\right)=\sum_{i=1}^{n} \beta_{i}\left(A_{i}\right)
$$

in which each factor contributes additively to total ambiguity. Here each $\beta_{i}$ maps $\mathscr{A}_{i}^{0}$ into $\mathbb{R}$, and it is natural to take $\beta_{i} \geq 0$ for all $i$ and extend the $\beta_{i}$ to the empty set by defining $\beta_{i}(\varnothing)=0$. The relationship of such $\beta_{i}$ to our marginal ambiguity functions is straightforward.

PROPOSITION 3. Suppose $\alpha$ is an ambiguity function that is additive on $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$ as $\alpha\left(A_{1}, \ldots, A_{n}\right)=\Sigma \beta_{i}\left(A_{i}\right)$ with $\beta_{i} \geqslant 0$ for each $i$. Define $\beta_{i}(\varnothing)=0$ for all $i$. Then $\beta_{i}$ is identical to the marginal functional $\alpha_{i}$ for each $i$.

Proof. Assume the hypotheses. Nonnegativity and $0=\alpha(S)=$ $\Sigma \beta_{i}\left(S_{i}\right)$ imply $\beta_{i}\left(S_{i}\right)=0$ for all $i$. Then, for $A_{i} \in \mathscr{A}_{i}^{0}$,

$$
\begin{aligned}
\alpha_{i}\left(A_{i}\right) & =\alpha\left(S_{1}, \ldots, S_{i-1}, A_{i}, S_{i+1}, \ldots, S_{n}\right) \\
& =\alpha\left(S_{1}, \ldots, A_{i}, \ldots, S_{n}\right)-\sum_{j \neq i} \beta_{j}\left(S_{j}\right) \\
& =\beta_{i}\left(A_{i}\right) .
\end{aligned}
$$

Since $\beta_{i}(\varnothing)=\alpha_{i}(\varnothing)=0$ by definition, $\beta_{i}=\alpha_{i}$.
It should be noted that if the additive form holds and the $\alpha_{i}$ or $\beta_{i}$ are given, we can obtain $\alpha$ only for $\varnothing$ and events in $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$. Although the additive form does not apply to the two-state event

$$
A=\left\{\left(s_{1}, \ldots, s_{n}\right),\left(t_{1}, \ldots, t_{n}\right)\right\}, \quad s_{i} \neq t_{i} \text { for some } i
$$

that form and submodularity give

$$
\begin{aligned}
\alpha(A) & \leqslant \alpha\left(\left\{\left(s_{1}, \ldots, s_{n}\right)\right\}\right)+\alpha\left(\left\{\left(t_{1}, \ldots, t_{n}\right)\right\}\right) \\
& =\sum_{i=1}^{n}\left[\alpha_{i}\left(\left\{s_{i}\right\}\right)+\alpha_{i}\left(\left\{t_{i}\right\}\right)\right] .
\end{aligned}
$$

More generally, for distinct $s^{j} \in S$, additivity and submodularity imply

$$
\alpha\left(\left\{s^{1}, \ldots, s^{m}\right\}\right) \leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i}\left(\left\{s_{i}^{j}\right\}\right) .
$$

An extension of the axiomatization in Section 2 suffices for the additive ambiguity representation. Assume the structure of the opening paragraph of the present section and define $P$ for strict ambiguity comparisons as before. Join to $N$ coefficient functions for additivity. That is, for each $\left(A_{1}, \ldots, A_{n}\right)$ for which at least two $A_{i}$ are proper subsets of their $S_{i}$, add to $N c$ and $c^{\prime}$ with nonzero values

$$
\begin{aligned}
& c\left(A_{1}, \ldots, A_{n}\right) \\
& \quad=1, \quad c\left(S_{1}, \ldots, A_{i}, \ldots, S_{n}\right)=-1 \text { when } A_{i} \subset S_{i} \\
& c^{\prime}\left(A_{1}, \ldots, A_{n}\right) \\
& \quad=-1, \quad c^{\prime}\left(S_{1}, \ldots, A_{i}, \ldots, S_{n}\right)=1 \text { when } A_{i} \subset S_{i} .
\end{aligned}
$$

The $\geqslant 0$ linear inequalities for these two amount to $\alpha\left(A_{1}, \ldots, A_{n}\right)=$ $\Sigma \alpha_{i}\left(A_{i}\right)$, the desired equality in view of Proposition 3. The augmented $N$ and the original $P$ are used in Axiom 2 to imply an $\alpha$ that represents $(\mathscr{A}, \gtrsim)$ and is additive over factors. The existence of such an $\alpha$ is guaranteed by Theorem 1 in Fishburn (1992b).

An additive ambiguity representation can have a tight uniqueness structure. Suppose each $\alpha_{i}$ codomain is a nondegenerate interval. The $\alpha_{i}$ are then unique up to multiplication by a positive scalar. Additivity implies the same thing for the $\alpha$ values of all events in $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$. If, in addition, every $A \in \mathscr{A}$ has an $\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right)$ in $\mathscr{A}_{1}^{0} \times \cdots \times A_{n}^{0}$ for which $\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right) \geqq A$, then $\alpha$ as a whole is unique up to multiplication by a positive scalar, or has a ratio scale.

One can also approach the additive ambiguity representation from the conjoint measurement perspective (Fishburn, 1970; Krantz et al. 1971; Wakker, 1989). I sketch this without giving all the structural details.

We begin with weak order, $A \gtrsim S$, and additivity and Archimedean axioms to obtain $\beta_{i}: \mathscr{A}_{i}^{0} \rightarrow \mathbb{R}$ for $i=1, \ldots, n$ such that $\beta_{i} \geqslant 0, \beta_{i}\left(S_{i}\right)=$ 0 and

$$
\left(A_{1}, \ldots, A_{n}\right) \gtrsim\left(B_{1}, \ldots, B_{n}\right) \Leftrightarrow \sum_{i=1}^{n} \beta_{i}\left(A_{i}\right) \geqslant \sum_{i=1}^{n} \beta_{i}\left(B_{i}\right) .
$$

With fixed origins for the $\beta_{i}$, axioms of preceding references imply that the $\beta_{i}$ are unique up to multiplication by a positive scalar. Assume this. Define $\alpha$ on $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$ by

$$
\alpha\left(A_{1}, \ldots, A_{n}\right)=\sum \beta_{i}\left(A_{i}\right) .
$$

Assume further that $A \sim A^{\mathrm{c}}$ and for each $A$ there is an $\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right)$ in $\mathscr{A}_{1}^{0} \times \cdots \times \mathscr{A}_{n}^{0}$ such that $A \sim\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right)$. Then extend $\alpha$ uniquely to all of $\mathscr{A}$ by taking $\alpha(A)=\alpha\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right)$.

This covers everything in the additive ambiguity representation except submodularity. In view of additivity and the solution axiom $A \sim\left(A_{1}^{\prime}, \ldots, A_{n}^{\prime}\right)$, when $A \backslash B \neq \varnothing$ and $B \backslash A \neq \varnothing$, submodularity requires

$$
\sum \beta_{i}\left(E_{i}\right)+\sum \beta_{i}\left(F_{i}\right) \leqslant \sum \beta_{i}\left(C_{i}\right)+\sum \beta_{i}\left(D_{i}\right)
$$

whenever

$$
\begin{aligned}
A & \sim\left(C_{1}, \ldots, C_{n}\right) \\
B & \sim\left(D_{1}, \ldots, D_{n}\right) \\
A \cap B & \sim\left(E_{1}, \ldots, E_{n}\right) \\
A \cup B & \sim\left(F_{1}, \ldots, F_{n}\right) .
\end{aligned}
$$

We can approach this axiomatically by a bisection or equal-spacing operation for each factor. For each $i$ let $Z_{i} \approx{ }_{i}\left(X_{i}, Y_{i}\right)$ for $X_{i}, Y_{i}$, $Z_{i} \in \mathscr{A}_{i}^{0}$ mean that there are $p$ and $q$ in the product of the $\mathscr{A}_{j}^{0}$ for $j \neq i$ such that

$$
\begin{aligned}
& \left(X_{i}, p\right) \sim\left(Z_{i}, q\right) \\
& \left(Z_{i}, p\right) \sim\left(Y_{i}, q\right)
\end{aligned}
$$

This implies that $\beta_{i}\left(Z_{i}\right)=\left[\beta_{i}\left(X_{i}\right)+\beta_{i}\left(Y_{i}\right)\right] / 2$. Assume that, given $X_{i}$ and $Y_{i}$, there are $Z_{i}, p$ and $q$ that satisfy the two $\sim$ expressions. Then, for the preceding instance of submodularity, we have $G_{i}, H_{i} \in \mathscr{A}_{i}^{0}$ for each $i$ such that

$$
G_{i} \approx{ }_{i}\left(C_{i}, D_{i}\right) \quad \text { and } \quad H_{i} \approx_{i}\left(E_{i}, F_{i}\right)
$$

In terms of the $\beta_{i}$ these translate into

$$
\begin{aligned}
& 2 \sum \beta_{i}\left(G_{i}\right)=\sum \beta_{i}\left(C_{i}\right)+\sum \beta_{i}\left(D_{i}\right) \\
& 2 \sum \beta_{i}\left(H_{i}\right)=\sum \beta_{i}\left(E_{i}\right)+\sum \beta_{i}\left(F_{i}\right)
\end{aligned}
$$

Hence the desired inequality for submodularity is tantamount to $\left(G_{1}, \ldots, G_{n}\right) \geq\left(H_{1}, \ldots, H_{n}\right)$.

In summary, our axiom for submodularity in the additive conjoint approach is

$$
\begin{aligned}
& \left(A \sim\left(C_{1}, \ldots, C_{n}\right), B \sim\left(D_{1}, \ldots, D_{n}\right), G_{i} \approx_{i}\left(C_{i}, D_{i}\right)\right. \\
& \quad \text { for all } i, \\
& A \cap B \sim\left(E_{1}, \ldots, E_{n}\right), A \cup B \sim\left(F_{1}, \ldots, F_{n}\right) \\
& \quad H_{i} \approx_{i}\left(E_{i}, F_{i}\right) \\
& \text { for all } i) \Rightarrow\left(G_{1}, \ldots, G_{n}\right) \geq\left(H_{1}, \ldots, H_{n}\right) .
\end{aligned}
$$

When this is adjoined to preceding axioms with suitable structural assumptions, we obtain conditions that are sufficient for the additive ambiguity representation with $\alpha$ unique up to multiplication by a positive constant. By Proposition 3, the nonnegative functions for factors in the additive part of the representation are the marginals of $\alpha$.

## 5. DISCUSSION

This paper continues a study (Fishburn, 1991) of event ambiguity as a primitive concept. Earlier axioms for finite event spaces were generalized to conditions on a comparative ambiguity relation on an arbitrary event space which imply the existence of a real valued function that preserves the ambiguity relationships and satisfies complementary equality and submodularity. We investigated uniqueness properties of such functions, then considered marginal ambiguity and additive decomposability over factors when the basic state space is the Cartesian product of $n$ other factors.

The work completed here raises questions that may interest others. The most basic is whether ambiguity really merits consideration as a primitive apart from more familiar concepts like comparative probability and choice. Another is the extent to which ambiguity can be investigated empirically without reference to likelihood or choice.

The present primitive approach has proposed axioms for comparative ambiguity that seem intuitively reasonable. Do some of them have unforeseen shortcomings? Are there other viable candidates for ambiguity axioms thus far overlooked?

Several technical matters might be pursued. One is to characterize more completely the family of ambiguity functionals that represent a qualitative ambiguity structure. Another asks whether there is a less abstract set of axioms than those in Section 2 for representability, perhaps under weak order and a rich structure for the event space in the infinite-events case.

## REFERENCES

Curley, S.P. and Yates, J.F.: 1989, 'An empirical evaluation of descriptive models of ambiguity reactions in choice situations', Journal of Mathematical Psychology 33, 397-427.
Einhorn, H.J. and Hogarth, R.M.: 1986, 'Decision making under ambiguity', Journal of Business 59 (2), S225-S250.
Ellsberg, D.: 1961, 'Risk, ambiguity, and the Savage axioms', Quarterly Journal of Economics 75, 643-669.
Fishburn, P.C.: 1970, Utility Theory for Decision Making, Wiley, New York.
Fishburn, P.C.: 1988, Nonlinear Preference and Utility Theory, Johns Hopkins University Press, Baltimore, MD.
Fishburn, P.C.: 1991, 'On the theory of ambiguity', International Journal of Information and Management Sciences 2 (2), 1-16.
Fishburn, P.C.: 1992a, 'On nonstandard nontransitive additive utility', Journal of Economic Theory 56, 426-433.
Fishburn, P.C.: 1992b, 'A general axiomatization of additive measurement with applications', Naval Research Logistics 39, 741-755.
Fishburn, P.C. and LaValle, I.H.: 1991, 'Nonstandard nontransitive utility on mixture sets', Mathematical Social Sciences 21, 233-244.
Hazen, G.B.: 1989, 'Ambiguity aversion and ambiguity content in decision making under uncertainty', Annals of Operations Research 19, 415-434.
Heath, C. and Tversky, A.: 1991, 'Preference and belief: ambiguity and competence in choice under uncertainty', Journal of Risk and Uncertainty 4, 5-28.
Herstein, I.N. and Milnor, J.: 1953, 'An axiomatic approach to measurable utility', Econometrica 21, 291-297.
Hogarth, R.M.: 1989, 'Ambiguity and competitive decision making: some implications and tests', Annals of Operations Research 19, 31-50.
Kelley, J.L.: 1955, General Topology, American Book Company, New York.
Keynes, J.M.: 1921, A Treatise on Probability, Macmillian, New York.
Knight, F.H: 1921, Risk, Uncertainty and Profit, Houghton Mifflin, Boston.
Krantz, D.H., Luce, R.D., Suppes, P. and Tversky, A.: 1971, Foundations of Measurement, Volume I, Academic Press, New York.
Narens, L.: 1985, Abstract Measurement Theory, MIT Press, Cambridge, MA.
Roberts, F.S.: 1979, Measurement Theory, Addison-Wesley, Reading, MA.
Savage, L.J.: 1954, The Foundations of Statistics, Wiley, New York.
Schmeidler, D.: 1989, 'Subjective probability and expected utility without additivity', Econometrica 57, 571-587.

Suppes, P., Krantz, D.H., Luce, R.D. and Tversky, A.: 1989, Foundations of Measurement, Volume II, Academic Press, San Diego.
Wakker, P.P.: 1989, Additive Representations of Preferences, Kluwer Academic, Dordrecht.
Wallsten, T.S.: 1990 , 'The costs and benefits of vague information', in: Hogarth, R.M. (ed.), Insights in Decision Making: A Tribute to Hillel J. Einhorn, University of Chicago Press, Chicago, pp. 28-43.
Wong, S.K.M., Yao, Y.Y., Bollmann, P. and Bürger, H.C.: 1991, 'Axiomatization of qualitative believe structure', IEEE Transactions on Systems, Man, and Cybernetics 21, 726-734.
Wong, S.K.M., Yao, Y.Y. and Lingras, P.: 1992, 'Comparative beliefs and their measurements', International Journal of General Systems (in press).

