Ambiguity, Trust and the Family

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Düsseldorf, 19.4.2023

Introduction to Ambiguology

It is generally thought that ambiguity makes sound reasoning difficult or impossible – our goal here is to prove three main claims:

1. We can reason perfectly well with ambiguity;
   but
2. strange things happen when we do so,
   and
3. there is not (and there cannot be) *the one* way to reason with ambiguity!

Reasoning with ambiguity inevitably leeds to logical pluralism (the family), most importantly based on whether we trust our interlocutor or not.

Outline

This talk is an introduction to and outline of the family of ambiguity logics, which can be split in two groups:

1. trustful
2. distrustful

It consists of three parts:

1. **Ambiguity**: basic properties of ambiguity, and fundamental results which apply to all logics which are supposed to treat the phenomenon adequatly
2. **Trust/Distrust**: we present a logic of mistrust and a logic of trust, and establish their basic properties.
3. **The Family**: we generalize these results and establish the family of amiguity logics, presenting more logics, methods of construction, and some interrelations between these logics.

Outline

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Preliminaries

(1) `bank` ('financial institute', 'strip of land along a river')

(2) `plant` ('fabric', 'thing making photosynthesis')

Old puzzle:

- If we treat ambuigity as a semantic phenomenon (i.e. assume the existence of *ambiguous meanings*),
- the question arises: what is the meaning of ambiguity?
- In particular, one has to account
  1. for certain peculiar combinatorial properties (*universal distribution*, see below), and
  2. some peculiar semantic properties (close to disjunction, but clearly distinct)

Prejudice 1: Ambiguity as disjunction

- We denote the ambiguity between two meanings by $\|$

$\|$ behaves differently than $\vee$ in a number of ways:

1. $\neg(a\|b) \equiv (\neg a)\|(\neg b)$, whereas $\neg(a \vee b) \equiv (\neg a) \wedge (\neg b)$
2. Assume $a \vdash b$. Then $a \vee b \equiv b$. But $a\|b \not\equiv b$ (`Every boy loves a movie`)
3. ...

Prejudice 1: Ambiguity as disjunction

Further differences:

(3) `I need the dough!`

($\equiv$ 'I need the pastry' $\|$ 'I need the money')

(4) `I need the pastry or I need the money!`

Denotational differences

- Uttering (4), I should be happy with either the pastry or the money.
- Uttering (3), I can reject the pastry and say: 'I intended the money, idiot' (or conversely reject the money).

So ambiguity comes with a different *committment/intention* than disjunction.

Prejudice 2: Humans disambiguate

- Often, (computational) linguists presume that humans disambiguate utterances before interpreting them

While this may *often* be the case, it is not generally true:

1. Evidence from semantic annotation shows that often meanings cannot be clearly identified (Poesio, various papers)
2. Immediate example: `The first thing that strikes a stranger in New York is a big car` etc.

Prejudice 3: Do it with meta-languages

- Often, (computational/formal) linguists use meta-languages, which 'interpret' ambiguity as underspecification (of the object language)

This might be convenient, does however not really address the issue:

1. The meta-languages usually do not satisfy the requirements of a semantic representation in the first place: well-defined semantics and a well-defined entailment relation
2. If we start studying these properties, the meta-languages become logics themselves – ambiguity logics!

Prejudice 4: Ambiguity is not semantic in nature

- There is no ambiguous word plant, but only $\texttt{plant}_1$, $\texttt{plant}_2$
- Similar with other phenomena (derivations, scope)

However, this comes short of reality:

1. The problem is that ambiguous meanings are **real** and we have to deal with them!
2. Every unresolved ambiguity would make comprehension impossible.
3. Disambigutation itself often takes place in semantic contexts, hence ambiguity has to come to semantics

(see also [6, 5, 7])

Intermediate summary

To sum up:

Reasoning with ambiguity is necessary for an adequate understanding and treatment of the phenomenon.

Note that disambiguation is just a particular instance of this:

$$(\alpha \| \beta \| \gamma) \wedge \neg\beta \vdash \alpha \| \gamma \tag{5}$$

There are many more inferences which are immediate:

$$\begin{aligned} \alpha \| \gamma &\vdash (\alpha \| \beta \| \gamma) \vee \neg\beta &(6)\\ \alpha \wedge \beta &\vdash \alpha \| \beta &(7)\\ \alpha \| \beta &\vdash \alpha \vee \beta &(8)\\ \dots & &(9) \end{aligned}$$

Intermediate summary

Note that reasoning with ambiguity is not only interesting from the perspective of 1. linguistics, but also for

2. Computer science (via natural language, ambiguity enters into ontologies, see [1])
3. Cognitive science
4. Philosophy (argumentation theory)

So let us have a closer look at the phenomenon!

Outline

Linguistic ambiguity

Linguistic ambiguity

Exponents of natural language give rise to two or more sharply distinguished meanings.

(10)
a. `bank` ('financial institute', 'strip of land along a river')
b. `kick the bucket` ('kick the bucket', 'die')
c. `see the man with the telescope` ('see with', 'man with')
d. `every boy loves a movie` ($\forall\exists$, $\exists\forall$)

Fundamental properties I: Universal Distribution

$$\neg(\alpha \| \beta) \equiv \neg\alpha \| \neg\beta \tag{11}$$

(12) a. `There is a bank`
b. `There is no bank`.

$$\alpha \wedge (\beta \| \gamma) \equiv (\alpha \wedge \beta) \| (\alpha \wedge \gamma) \tag{13}$$
$$\alpha \vee (\beta \| \gamma) \equiv (\alpha \vee \beta) \| (\alpha \vee \gamma) \tag{14}$$

(15) `I love plants and mushrooms` etc.

Universal Distribution

All connectives uniformly distribute over $\|$. Hence every formula has an ambiguous normal form $a_1 \| ... \| a_i$. Note the potential exponential growth!

Fp II: Unambiguous entailments from ambiguous statements

$$\alpha \wedge \beta \vdash \alpha \| \beta \vdash \alpha \vee \beta \tag{16}$$

(17) `Money and pastry` entails dough entails `money or pastry`

Note that there is a complete symmetry: $\alpha \| \beta$ is as close to $\alpha \vee \beta$ as it is to $\alpha \wedge \beta$.

Fp III: Conservative extension

This is a property which is almost too obvious:

Principle of conservative extension

A logic extended with ambiguous proposition should *not* allow for any additional entailments on unambiguous propositions.

Put differently: the presence of ambiguity does not affect the entailments between unambiguous formulas.

Goes without further comment.

Fp IV: Associativity

$$(\alpha \| \beta) \| \gamma \vdash \alpha \| (\beta \| \gamma) \tag{18}$$

Put differently: the subparts of an ambiguous meaning do not come in groupings.

Goes without further comment.

Questionable property I: Law of Disambiguation

$$(\alpha \| \beta \| \gamma) \wedge \neg \beta \vdash \alpha \| \gamma \tag{19}$$

This seems convincing, but as a matter of fact, many ambiguity logics (of distrust) do not satisfy this general version: they require that β itself is unambiguous.

$$(\alpha \| \boldsymbol{b} \| \gamma) \wedge \neg \boldsymbol{b} \vdash \alpha \| \gamma, \quad \text{for b unambiguous} \tag{20}$$

And we will even see one (unreasonable) ambiguity logic which does not satisfy this, hence I prefer not to include this into the list...

We call 19 the **general** law, 20 the **special** law of disambiguation.

Questionable property II: Monotonicity

$$\alpha \vdash \alpha' \;\&\; \beta \vdash \beta' \implies \alpha \| \beta \vdash \alpha' \| \beta' \tag{21}$$

So:

(22) `plants entails plants or animals`

This seems convincing too, but not necessary.

It actually entails the weak law of disambiguation, hence we see this as a reasonable property, but not a necessary one.

Questionable property III: Commutativity

$$\alpha \| \beta \equiv \beta \| \alpha \tag{23}$$

This is an arguable property which not all logics satisfy: often, readings do have an intrinsic ordering (primary, secondary, ...)

Commutativity is not at all necessary (and not always desirable), but important to separate various ambiguity logics.

Questionable property IV: Consistent usage

This is another important property to separate classes of logics.

Uniform usage (UU)

In a given context, an ambiguous statement is used consistently in **only one** sense.

- In many cases, this is obviously wrong, if context determines different readings

  (24) `I spring over a spring in spring`

- On the other hand, in many other cases this is reasonable to assume (if I talk about `plants`, you should presume I do so for one reading, see [8]).

This property is what separates (conceptually) trustful and distrustful logics of ambiguity.

An illustration

To illustrate how UU and Monotonicity impact formal logic:

- If a logic does not assume (UU), then

$$(\alpha \| \beta) \wedge \neg(\alpha \| \beta) \tag{25}$$

is usually **not** a contradiction.

- If a logic does not assume Monotonicity, then even

$$\alpha \| \beta \vdash \alpha \| \beta \tag{26}$$

might **not** be derivable!

Outline

Trigger warning

So we have some properties which are correct beyond reasonable doubt. But:
These properties, together with the usual properties of classical logic, allow for inferences which are "beyond" our intuition.
Example?

$$\begin{aligned}
\alpha\|\beta &\equiv (\alpha\|\beta) \wedge (\alpha\|\beta) && (27)\\
&\equiv (\alpha \wedge (\alpha\|\beta))\|(\beta \wedge (\alpha\|\beta)) && (28)\\
&\equiv \ldots && (29)\\
&\equiv \alpha\|(\alpha \wedge \beta)\|\beta && (30)\\
& \ldots && (31)
\end{aligned}$$

This can be extended and continued in various ways!

The main goal

Let us (re-state) the main goal:

The main goal

We want a **formal logic** which

1. Conservatively extends classical logic
2. Derives all and only valid inferences for ambiguous statements, in particular it respects
   - 2.1 (mandatory) the laws of universal distribution
   - 2.2 (mandatory) the laws of unambiguous entailments
   - 2.3 (mandatory) associativity of ambiguity
3. and possibly it satisfies
   - 3.1 (facultative) the law of dismbiguation
   - 3.2 (facultative) the law of monotonicity
   - 3.3 (facultative) the law of commutativity of ambiguity
   - 3.4 (facultative) the law of consistent usage

Now comes the fundamental theorem:

The fundamental theorem, algebraic version

This comes in various versions; the basic version is algebraic:

Theorem

Let $\mathbf{A} = (A; \wedge, \vee, \sim, 0, 1, \|)$ *be an algebra such that*

- $(A; \wedge, \vee, \sim, 0, 1)$ *is a Boolean algebra*
- $\|$ *satisfies the laws of*
  1. *Universal distribution*
  2. *Unambiguous entailments*
  3. $\|$*-associativity*

Then for all $a, b, c \in A$, $\mathbf{A}$ *satisfies* $a\|c\|b = a\|b$

Corollary

Assume $\mathbf{A}$ *in addition satisfies* $a\|b = b\|a$*. Then* $\mathbf{A}$ *is trivial, i.e. one element.*

The fundamental theorem, algebraic version

This means no less than the following:

Meaning of the fundamenal theorem

Every algebra which satisfies the most basic requirements of ambiguity already derives equalities which are strongly counter-intuitive.

Consequence

Algebra itself is fundamentally inapt to treat the phenomenon of ambiguity. This inaptness can be tracked down to two intrinsic properties of all algebras:

1. Assume $t_1[t] = t_2$, $t = t'$ holds (is valid) in a class of algebras. Then $t_1[t'] = t_2$ holds as well.
2. Assume $t_1 = t_2$ holds (is valid) in a class of algebras, σ is a map from atoms to arbitrary terms. Then $\sigma(t_1) = \sigma(t_2)$ holds as well.

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2. Assume $t_1 = t_2$ holds (is valid) in a class of algebras, σ is a map from atoms to arbitrary terms. Then $\sigma(t_1) = \sigma(t_2)$ holds as well.

Property 1 means: equal terms can be substituted in all contexts, preserving validity of equations. We call this **e-substitution**.

Property 2 means: atoms can be uniformly substituted, preserving validity of equations. We call this **u-substitution**.

An example from arithmetic

e-congruence:

$$\begin{aligned} (4+5)\cdot 3 &= 27 \\ 2+3 &= 5 \\ \therefore (4+(2+3))\cdot 3 &= 27 \end{aligned}$$

u-congruence:

$$\begin{aligned} (x+y)\cdot z &= x\cdot z + y\cdot z \\ x &\mapsto z+x \\ \therefore ((z+x)+y)\cdot z &= (z+x)\cdot z + y\cdot z \end{aligned}$$

The fundamental theorem, logical version

Theorem

Let $\mathcal{L} = (Form, \vdash)$ be an ambiguity logic which

- *conservatively extends classical logic*
- *satisfies the laws of*
    1. *universal distribution*
    2. *Unambiguous entailments*
    3. *$\|$-associativity*

Moreover, assume LA *allows the rule (cut) and is closed under uniform substitution. Then for all α, β, γ,* LA *derives*
$\alpha\|\gamma\|\beta \dashv\vdash \alpha\|\beta$

Corollary

Assume LA *in addition satisfies $\alpha\|\beta \vdash \beta\|\alpha$. Then* LA *is inconsistent, i.e. derives everything.*

The fundamental theorem, logical version

Meaning of the fundamenal theorem, logically

Every non-trivial logic of ambiguity either

- is not closed under uniform substitution, meaning: entailments are not generally valid schemes, validity depends on instantiations; or
- is not closed under e-substitution, meaning: being logically equivalent does *not* imply being exchangeable in all contexts. In practice, these logics are usually not even transitive!

Note that in both cases, we do not satisfy properties which are considered fundamental for *all logical consequence relations* in the tradition of abstract logics according to Tarski!

The fundamental theorem, consequence

Consequence of the fundamenal theorem

There are four types of ambiguity logics, two of which are relevant:

1. closed under e-substiution and u-substitution: trivial
2. closed under neither e- nor u-substitution: uninteresting? (why would anyone do this?)
3. closed under e-substitution, not u-substitution: **distrustful logics**.
4. closed under u-substitution, not e-substitution: **trustful logics**.

We will have a closer look at 3. and 4.

Distrustful logics

3. closed under e-substitution, not u-substitution: **distrustful logics**.

Why are these logics distrustful? The intuition is:

Intuition of distrustful logics

If I want to know whether an argument holds (e.g. $\vdash \alpha \vee \neg\alpha$), I need to check all involved propositions. I cannot trust the scheme!

For example:

$$\begin{array}{ll} \vdash & p \vee \neg p \\ \nvdash & (p \| q) \vee \neg(p \| q) \end{array}$$

(32) `He is dead and he is not dead.`

Trustful logics

4. closed under u-substitution, not e-substitution: **trustful logics**.

Why are these logics trustful? The intuition is:

Intuition of trustful logics

All inferences are valid as schemes, regardless of their internal content!

Note: non-trivial trustful ambiguity logics are *never* closed under e-substutition! How can we conceptually make sense of this?

Trustful logics

Note: non-trivial trustful ambiguity logics are *never* closed under e-substutition! How can we make sense of this?

The answer lies (maybe) in the fact that syntax and semantics cannot be very neatly separated:

(33) `Ich heiße Heinz Erhard und Sie herzlich Willkommen`

(34) `She made no reply, up her mind, and a dash for the door.`

In order to properly reason, we need to remember exact semantic form, not only inferential properties (tentative).

Trustful logics

Note: non-trivial trustful ambiguity logics are *never* closed under e-substutition! How can we make sense of this?

See the following example:

$$\begin{array}{rcl} (p\|q) \vee \neg(p\|q) & \equiv & (p \vee \neg p)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q) \\ & \not\equiv & (r \vee \neg r)\|(p \vee \neg q)\|(\neg p \vee q)\|(q \vee \neg q) \end{array}$$

In this example, $p \vee \neg p$ is *not* (necessarily) an arbitrary theorem – the variable p is linked to its other occurrences!

Trustful logics: Transitivity

A special case of closure under e-substitution is *transitivity*.

- Consequently, trustful ambiguity logics are usually not transitive!

We can make sense of this as follows:

(35) a. `Ice is water.`
b. `Water is liquid.`
c. `Ice is liquid.`

a.,b. might be acceptable under very trustful circumstances (*cTAL* below), c. is unacceptable under any.

Hence transitivity cannot be generally permitted!

Side note: natural language is not closed under e-substitution

Propositional attitudes

- She believes that P
- She thinks that P
- She says that P

(36) a. In PA, $2 + 2 = 4$
b. In PA, $2 + 2 = 5$
c. In PA, the strong Goldbach conjecture is true.

Now since the (open) Goldbach conjecture is either true or false, it follows that:

(36-c) is either equivalent to (36-a) or (36-b).

Side note: natural language is not closed under e-substitution

But the statements are obviously not interchangeable in all contexts:

(37) a. She believes that in PA $2+2=4$
b. She believes that in PA $2+2=5$
c. She believes that in PA the strong Goldbach conjecture is true.

But: (38) is exchangeable with (36-b):

(38) She believesthat in PA $2+2$ is 1 more than 4.

- Logical equivalence does not mean exchangeability in all contexts
- Some are interchangeable, others are not!
- Hence this property is exotic from a logical point of view, not a linguistic one!

Outline

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Two logics

We now present two logics, very different in nature:

1. DAL, a (commutative) logic of distrust. It is most naturally conceived of semantically.
2. TAL, a non-commutative logic of trust. It is most naturally conceived of proof-theoretically.

Both logics are well-investigated and seem to be very reasonable. Despite their differences, we will later on see that they are connected in several interesting ways!

Distrustful Ambiguity Logic DAL

We now present the logic DAL

- DAL ist not closed under u-substitution.
- DAL has several very good independent definitions and motivations
- Discovered van Eijck, Jaspars ([3]): Reasoning and Ambiguity.

Distrustful Ambiguity Logic DAL

The language is obviously the following (as in all of our logics, propositional logic with $\|$):

- If $p \in \texttt{Var}$, then $p \in \texttt{WFF}$;
- if $\alpha, \beta \in \texttt{WFF}$, then $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \| \beta), (\alpha \rightarrow \beta), (\neg \alpha) \in \texttt{WFF}$;
- nothing else is in $\texttt{WFF}$.

(I usually omit $\rightarrow$, since it is interdefinable)

Uniform Substitution

Assume $\sigma : \mathtt{Var} \to \mathtt{WFF}$ is a function (variables to formulas). This is extended to a uniform substitution

- $\sigma(p) = \sigma(p)$, $p \in \mathtt{Var}$
- $\sigma(\alpha \wedge \beta) = \sigma(\alpha) \wedge \sigma(\beta)$
- $\sigma(\alpha \vee \beta) = \sigma(\alpha) \vee (\beta)$
- $\sigma(\alpha \to \beta) = \sigma(\alpha) \to \sigma(\beta)$
- $\sigma(\neg\alpha) = \neg\sigma(\alpha)$

Every normal logic is closed under uniform substitution.

DAL

Recall the following:

$$\alpha \wedge \beta \models \alpha \| \beta \models \alpha \vee \beta.$$

From here, we can (semantically) draw two conclusions:

1. If $M \models \alpha \wedge \beta$, then $M \models \alpha \| \beta$
2. If $M \not\models \alpha \vee \beta$, then $M \not\models \alpha \| \beta$

DAL is based on the observation that the dualism true/non true is not enough to adequatly treat ambiguity.

Verification and Falsification (Van Eijck,Jaspars)

Hence we make additional distinctions:

- A model **verifies** a formula,
- A model **does not** verify a formula,
- A model **falsifies** a formula,
- A model **does not** falsify a formula

These notions do *not* coincide. Hence we need

- Two relations $\models$, $\rightsquigarrow$, as well as
- their negations $\not\models$, $\not\rightsquigarrow$

Verification and Falsification (Van Eijck,Jaspars)

For formulas α of classical logic, we obviously have

$$M \not\models \alpha \text{ iff } M \rightsquigarrow \alpha$$
$$M \not\rightsquigarrow \alpha \text{ iff } M \models \alpha$$

But this does not extend to ambiguous formulas!

Essence of ambiguity in DAL

Ambiguous formulas can be not strictly true and not strictly false.

Verification and Falsification: An example

Take 3 classical models:

1. $M_1 = \emptyset$
2. $M_2 = \{p\}$
3. $M_3 = \{p, q\}$

Take the formula $p \| q$. We obtain

1. $M_1 \leadsto p\|q$ (since $M \not\models p \vee q$)
2. $M_2 \not\leadsto p\|q$, $M_2 \not\models p\|q$
3. $M_3 \models p\|q$ (since $M \models p \wedge q$)

The following always holds:

$$M \leadsto \alpha \text{ implies } M \not\models \alpha$$
$$M \models \alpha \text{ implies } M \not\leadsto \alpha$$

Verification and Falsification

(As a side note: this gives rise to a Kleene-style three-valued logic:

1. α is true in a model
2. α is false in a model
3. α is neither true nor false in a model

We will not spell this out, but this is another way to define DAL.)
Instead, we define truth and falsity:

Verification and falsification in *DAL*

$\models$1 $M \models p_i$ iff $p_i \in M$.

$\models$2 $M \models \alpha \wedge \beta$, iff $M \models \alpha$ and $M \models \beta$.

$\models$3 $M \models \alpha \vee \beta$, iff at least one holds, $M \models \alpha$ or $M \models \beta$.

$\models$4 $M \models \neg\alpha$, iff $M \rightsquigarrow \alpha$ (*not* $M \not\models \alpha$!!)

$\models$5 $M \models \alpha \| \beta$, iff $M \models \alpha \wedge \beta$.

So: regarding $\models$, $\|$ behaves like $\wedge$!

Verification and falsification in *DAL*

$\rightsquigarrow$1 $M \rightsquigarrow p_i$ iff $p_i \notin M$

$\rightsquigarrow$2 $M \rightsquigarrow \alpha \wedge \beta$, iff at least one holds, $M \rightsquigarrow \alpha$ or $M \rightsquigarrow \beta$

$\rightsquigarrow$3 $M \rightsquigarrow \alpha \vee \beta$, iff $M \rightsquigarrow \alpha$ and $M \rightsquigarrow \beta$

$\rightsquigarrow$4 $M \rightsquigarrow \neg\alpha$, iff $M \models \alpha$

$\rightsquigarrow$5 $M \rightsquigarrow \alpha \| \beta$, iff $M \rightsquigarrow \alpha \vee \beta$.

So: regarding $\rightsquigarrow$, $\|$ behaves like $\vee$!

Logical consequence in DAL

We have defined two relations between models and formulas

We now want a **single relation** between formulas.

We define $\Rightarrow_{DAL}$ as follows:

Logical consequence in DAL: $\Rightarrow_{DAL}$

1. Write $\alpha \models \beta$ iff $M \models \alpha$ implies $M \models \beta$
2. Write $\alpha \rightsquigarrow \beta$ iff $M \rightsquigarrow \alpha$ implies $M \rightsquigarrow \beta$
3. $\alpha \Rightarrow_{DAL} \beta$ holds iff $\alpha \models \beta$ und $\beta \rightsquigarrow \alpha$.

Logical consequence in DAL

Hence logical consequence in DAL is defined by preservation of truth and inverse preservation of falsity.

This definition by Van Eijck, Jaspars is intuitive, but clumsy to use in practice.

There is an equivalent definition which makes life much easier:

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Alternative formulation

We can use the following notions:

- $M \models_{\Box} \alpha$, if α is **necessarily** true in M
- $M \models_{\Diamond} \alpha$, if α is **possibly** true in M

Now assume α, β are unambiguous. Then we obtain:

1. $M \models_{\Box} \alpha \| \beta$ iff $M \models \alpha \wedge \beta$
2. $M \models_{\Diamond} \alpha \| \beta$ iff $M \models \alpha \vee \beta$

Alternative formulation

This provides the following equivalence:

Lemma

$M \models_{\Box} \alpha$ iff $M \models \alpha$
$M \models_{\Diamond} \alpha$ iff $M \not\rightsquigarrow \alpha$
$M \not\models \alpha$ iff $M \not\models_{\Box} \alpha$
$M \rightsquigarrow \alpha$ iff $M \not\models_{\Diamond} \alpha$

Alternative formulation

But: instead of using the relations $\models_{\Box}$ etc., it is much more convenient to modify formulas. We define $\blacksquare, \blacklozenge : \mathtt{WFF} \to \mathtt{WFF}$.

Definition

$$\begin{array}{rcl}
\blacksquare(p) & = & p \\
\blacksquare(\alpha \wedge \beta) & = & \blacksquare(\alpha) \wedge \blacksquare(\beta) \\
\blacksquare(\alpha \vee \beta) & = & \blacksquare(\alpha) \vee \blacksquare(\beta) \\
\blacksquare(\neg\alpha) & = & \neg\blacklozenge(\alpha) \\
\blacksquare(\alpha \| \beta) & = & \blacksquare(\alpha) \wedge \blacksquare(\beta)
\end{array}$$

Alternative formulation

Dually:

Definition

$$
\begin{aligned}
\blacklozenge(p) &= p \\
\blacklozenge(\alpha \wedge \beta) &= \blacklozenge(\alpha) \wedge \blacklozenge(\beta) \\
\blacklozenge(\alpha \vee \beta) &= \blacklozenge(\alpha) \vee \blacklozenge(\beta) \\
\blacklozenge(\neg\alpha) &= \neg\blacksquare(\alpha) \\
\blacklozenge(\alpha \| \beta) &= \blacklozenge(\alpha) \vee \blacklozenge(\beta)
\end{aligned}
$$

Alternative formulation

Theorem

$\alpha \Rightarrow_{DAL} \beta$ if and only if (in classical logic): $\blacksquare\alpha \models \blacksquare\beta$ and $\blacklozenge\alpha \models \blacklozenge\beta$

Hence $\Rightarrow_{DAL}$ can be, via a simple transformation, reduced to classical logic.

This simple result allows us to prove a huge number of results.

Results on DAL

Positive results:

1. DAL conservatively extends classical logic
2. DAL satisfies universal distribution and unambiguous entailments
3. DAL is closed under e-substitution
4. DAL is obviously commutative

Negative results:

1. DAL is not closed under u-substitution
2. DAL does not satisfy Modus ponens (general version), only the special version
3. DAL does not satisfy the general law of disambiguation, only the special version

Results on DAL

There are many more results on DAL. But we sum up for now:

- DAL is a very natural and reasonable logic of distrust!

Now we present a trustful logic.

Outline

Multi-sequents

To present the proof-theory for DAL, we need slightly more complex structures which we call multi-contexts.

- In classical logic, $\alpha_1, ..., \alpha_i \vdash \beta_1, ..., \beta_j$ means: $\alpha_1 \wedge ... \wedge \alpha_i \vdash \beta_1 \vee ... \vee \beta_j$
- We need a context to introduce $\|$
- Hence: $\alpha_1; \alpha_2 \vdash \beta_1; \beta_2$, which means: $\alpha_1 \| \alpha_2 \vdash \beta_1 \| \beta_2$
- Careful with bracketing!

TAL: Gentzen calculus

Here the proof rules:

$$(\mathrm{ax})\ \frac{}{\alpha, \Gamma \vdash \alpha, \Delta}$$

$$(\wedge \mathsf{l})\ \frac{\Gamma[\alpha, \beta] \vdash \Theta}{\Gamma[\alpha \wedge \beta] \vdash \Theta} \qquad (\mathsf{l}\wedge)\ \frac{\Gamma \vdash \Theta[\alpha] \quad \Gamma \vdash \Theta[\beta]}{\Gamma \vdash \Theta[\alpha \wedge \beta]}$$

$$(\vee \mathsf{l})\ \frac{\Gamma[\alpha] \vdash \Theta \quad \Gamma[\beta] \vdash \Theta}{\Gamma[\alpha \vee \beta] \vdash \Theta} \qquad (\mathsf{l}\vee)\ \frac{\Gamma \vdash \Theta[\alpha, \beta]}{\Gamma \vdash \Theta[\alpha \vee \beta]}$$

TAL: Gentzen calculus

Weakening and contraction are admissible, we only need: (,comm)

$$(\text{,comm})\ \frac{\Gamma[\Psi,\Theta]}{\Gamma[\Theta,\Psi]}. \qquad (\text{,weak})\ \frac{\Gamma[\Delta]}{\Gamma[\Delta,\Psi]} \qquad (\text{,contr})\ \frac{\Gamma[\Delta,\Delta]}{\Gamma[\Delta]}$$

(Notation means: on both sides of $\vdash$)

Note: Multicontexts are not sets, but terms!

TAL: Gentzen calculus

Introduction of ; generalizes monotonicity,unambiguous entailments:

$$\text{(I;I)}\ \frac{\Gamma, \Lambda \vdash \Delta, \Psi \quad \Theta, \Lambda \vdash \Phi, \Psi}{(\Gamma; \Theta), \Lambda \vdash g(\Delta; \Phi), \Psi}$$

This generalizes several simpler rules:

$$(;1)\ \frac{\Gamma \vdash \Theta \quad \Delta \vdash \Phi}{(\Gamma; \Delta) \vdash (\Theta; \Phi)} \qquad (;2)\ \frac{\Gamma \vdash \Theta \quad \Delta \vdash \Theta}{(\Gamma; \Delta) \vdash \Theta} \qquad (;3)\ \frac{\Theta \vdash \Gamma \quad \Theta \vdash \Delta}{\Theta \vdash (\Gamma; \Delta)}$$

TAL: Gentzen calculus

The rules for $\|$ are now very simple:

$$(\|\mathsf{l})\ \frac{\Gamma[(\alpha;\beta)] \vdash \Theta}{\Gamma[\alpha\|\beta] \vdash \Theta} \qquad (\mathsf{l}\|)\ \frac{\Gamma \vdash \Theta[(\alpha;\beta)]}{\Gamma \vdash \Theta[\alpha\|\beta]}$$

TAL: Gentzen calculus

(;contr) is admissible (so we do not actually need it).

$$
(;\mathrm{ass})\ \frac{\Psi[(\Gamma;(\Delta;\Theta))]}{\Psi[((\Gamma;\Delta);\Theta))]} \qquad\qquad (;\mathrm{contr})\ \frac{\Gamma[(\alpha;\alpha)]}{\Gamma[\alpha]}
$$

DAL: negation

Negation in DAL falls into three rules. The classical rule

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \Delta, \neg\alpha}$$

is **not** sound in DAL. Instead we have:

$$\frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \qquad \frac{\Gamma, \neg p \vdash \Delta}{\Gamma \vdash \Delta, p} \ (p \text{ atomic}) \qquad \frac{\Gamma \vdash \Delta}{\neg\Gamma \vdash \neg\Delta} \ (\text{contra})$$

$$\frac{\Gamma, \neg\neg\alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \neg\neg\alpha}{\Gamma \vdash \Delta, \alpha} \ (\text{DN-elimination})$$

(The inverse is derivable)

TAL: cut

Finally, we have (cut), which has a very peculiar role

$$\text{(cut)}\ \frac{\Gamma[\alpha] \vdash \Psi \quad \Delta \vdash \alpha, \Theta}{\Gamma[\Delta] \vdash \Psi, \Theta} \qquad\qquad \text{(simple cut)}\ \frac{\Gamma[\alpha] \vdash \Psi \quad \Delta \vdash \alpha,}{\Gamma[\Delta] \vdash \Psi}$$

Cut is part of DAL, but the rule is not admissible (which means: we need it)!

DAL – $\|$-commutativity

$$(;\text{com})\ \frac{\Gamma[(\Psi;\Delta)]}{\Gamma[(\Delta;\Psi)]}$$

This rule ensures $\|$-commutativity. Without this rule, we obtain $\overline{c}DAL$, the non-commutative fragment, which

- is very interesting, but
- does not have a simple truth-theoretic semantics (obviously)

Outline

TAL and negation

The main motivation for this presentation was: we can easily get to TAL, the trustful ambiguity logic. We need only three changes:

1. Instead of three restricted negation rules, we take one generalized negation (which subsumes all three and classical negation)
2. (cut) is no longer part of the calculus
3. but to compensate, we need two additional rules

TAL: negation

We simply generalize the classic rule:

$$(\neg\mathsf{I})\ \frac{\Gamma \vdash \Delta, (\alpha_1; ...; \alpha_i)}{\Gamma, (\neg\alpha_1; ...; \neg\alpha_i) \vdash \Delta} \qquad (\mathsf{I}\neg)\ \frac{\Gamma, (\alpha_1; ...; \alpha_i) \vdash \Delta}{\Gamma \vdash \Delta, (\neg\alpha_1; ...; \neg\alpha_i)}$$

For $i = 1$, this is classic negation. This ensures distribution of $\neg$ over $\|$.

These two not sound in *DAL*!

TAL without cut

If we do not admit (cut), we need the following rules to ensure distribution and invertibility:

$$(\text{distr})\ \frac{\Gamma[_{(}\Delta;\Psi),\Theta_1] \quad \Gamma[_{(}\Delta;\Psi),\Theta_2]}{\Gamma[(((\Delta,\Theta_1);(\Psi,\Theta_2))]}$$

$$(\text{subst})\ \frac{\Gamma[\Psi] \quad \Gamma[(\Delta;\beta;\Delta')]}{\Gamma[(\Delta;\Psi;\Delta')]}$$

TAL (without cut): (distr)

$$
(\mathrm{distr})\ \frac{\Gamma[_{(}\Delta;\Psi),\Theta_1] \quad \Gamma[_{(}\Delta;\Psi),\Theta_2]}{\Gamma[(((\Delta,\Theta_1);(\Psi,\Theta_2)]}
$$

That is a rule for distribution: put $\Theta_1 = \Theta_2 = \Theta$. Then we obtain

$$
\frac{\Gamma[_{(}\Delta;\Psi),\Theta] \vdash \Xi}{\Gamma[(((\Delta,\Theta);(\Psi,\Theta)] \vdash \Xi}
$$

TAL (without cut): (subst)

$$\text{(subst)}\ \frac{\Gamma[\Psi] \quad \Gamma[(\Delta; \beta; \Delta')]}{\Gamma[(\Delta; \Psi; \Delta')]}$$

Assume, $\Gamma[\psi] \vdash \Xi$ und $\Gamma[(\delta; \beta; \delta')] \vdash \Xi$

Then: $\Gamma[\delta \| \beta \| \delta'] \vdash \Xi$

Hence: $\Gamma[(\delta \| \beta \| \delta') \vee \psi] \vdash \Xi$

Hence: $\Gamma[(\delta \vee \psi) \| (\beta \vee \psi) \| (\delta' \vee \psi)] \vdash \Xi$

Hence: $\Gamma[\delta \| \psi \| \delta'] \vdash \Xi$

This is an instance of (subst)!

TAL: Conventions

- The logic without (cut),(;com) but with (distr) and (subst) we call TAL, with (;com) cTAL
- The logic with (cut) we call $\text{TAL}^{cut}/\text{cTAL}^{cut}$. With (cut), (distr) and (subst) are admissible.
- The Logic without (cut) and (distr) is not an ambiguity logic.
- The Logic without (cut) and (subst) lacks certain invertibility properties.

Observation
TAL, TAL^{cut} are closed under substitution.

Application of the Fundamental Theorem
TAL^{cut} is trivial. cTAL^{cut} is inconsistent.

TAL and DAL

To understand the relation of TAL and DAL, consider the following:

- All rules of TAL are sound in DAL, with the exception of (¬I),(I¬).
- Conversely, the rule (cut) is sound in DAL.
- DAL is weaker than *cTAL* (not *TAL*), and $\overline{c}$*DAL* is weaker than *TAL*.

Incongruence

However, the central property of TAL,cTAL is that they are **incongruent**: $\alpha \dashv\vdash \beta$ does *not* entail $\Gamma[\alpha] \vdash \Delta$ iff $\Gamma[\beta] \vdash \Delta$.

(In)congruence

Incongruence

However, the central property of TAL,cTAL is that they are **incongruent**: $\alpha \dashv\vdash \beta$ does *not* entail $\Gamma[\alpha] \vdash \Delta$ iff $\Gamma[\beta] \vdash \Delta$ etc.

Hence derivability ($\vdash$) is not even the most important relation between formulas:

Definition of $\leqq_{TAL}$

We define $\alpha \leqq_{TAL} \beta$, if $\Gamma[\beta] \vdash \Delta$ entails $\Gamma[\alpha] \vdash \Delta$, and $\Gamma \vdash \Delta[\alpha]$ entails $\Gamma \vdash \Delta[\beta]$. Same for cTAL.

He $\leqq$ is the relation of "being logically stronger in all context", which does *not* coincide with entailment in this case!

Inner and outer logics

An example?

Because of closure under u-substitution, extension of classical logic, every trustful logic derives

$$\vdash (p \| q) \vee \neg (p \| q) \tag{39}$$

Because of universal distribution, we derive

$$\vdash (p \vee \neg p) \| (p \vee \neg q) \| (q \vee \neg p) \| (q \vee \neg q) \tag{40}$$

However: if we substitute $p \vee \neg p$ with $r \vee \neg r$, the result is not necessarily a theorem:

$$\vdash (r \vee \neg r) \| (p \vee \neg q) \| (q \vee \neg p) \| (q \vee \neg q) \tag{41}$$

is *not* derivable in every trustful ambiguity logic.

Some results on TAL

We can easily prove the following (positive) results:

- TAL,cTAL derives Modus ponens (general form): $\alpha, \alpha \rightarrow \beta \vdash \beta$
- TAL,cTAL derives the strong law of disambiguation: $\alpha \| \beta \| \gamma, \neg\beta \vdash \alpha \| \gamma$
- TAL,cTAL derive universal distribution
- and unambiguous entailments: $\alpha \wedge \beta \vdash \alpha \| \beta \vdash \alpha \vee \beta$
- closure under u-substitution means that all inferences are schematically valid.

Some results on TAL

There are some surprising results:

- TAL,cTAL are not closed under e-substitution. Even more: they are *not* transitive:
- there are α, β, γ such that $\alpha \vdash \beta$, $\beta \vdash \gamma$, but $\alpha \nvdash \gamma$!

There is an even stronger result:

Lemma
The transitive closure of TAL is trivial. The transitive closure of cTAL is inconsistent.

Hence there is little margin for extension!

The inner logic

Observation 1

$\leqq_{TAL}$, $\leqq_{cTAL}$ are transitive by definition.

Observation 2

This entails that

$$\leqq_{TAL} \subsetneq \vdash_{TAL} \tag{42}$$

$$\leqq_{cTAL} \subsetneq \vdash_{cTAL} \tag{43}$$

since otherwise this would result in trivial logics (Fundamental Theorem)

In fact, these relations have remained mysterious to me for quite some time. So I am happy to state the following result, connecting the *most important trustful* and *distrustful logics*:

The inner logic

Theorem

$\alpha \leqq_{cTAL} \beta$ *if and only if* $\alpha \Rightarrow_{DAL} \beta$

This is a very surprising connection, having two corollaries:

Corollary

1. $\leqq_{TAL} \subseteq \leqq_{cTAL}$
2. $\leqq_{TAL}$ *conservatively extends classical logic.*

Hence the most important distrustful logic is the inner logic of the most important trustful logic!!

The inner logic

Conjecture

$\alpha \leqq_{TAL} \beta$ if and only if $\alpha \Rightarrow_{\overline{c}DAL} \beta$, where $\overline{c}DAL$ is the non-commutative fragment of DAL.

Only if is rather straightforward (induction on proof length, soundness).

$\alpha \Rightarrow_{\overline{c}DAL} \beta$ entails $\alpha \leqq_{TAL} \beta$ is hard – there is nothing to induce over!

Intermediate Summary Trust

To sum up:

1. DAL is a reasonable commutative logic of mistrust
2. TAL is a reasonable non-commutative logic of trust
3. cTAL is an unreasonably liberal logic of trust (too naive), as we will see later
4. $\overline{c}DAL$ might be reasonable, but we know little of it...

There are of course many more results on TAL,cTAL and DAL. However, at this point we can go a step beyond and

define the family of ambiguity logics!

Outline

Outline

Meet the family

What is an ambiguity logic? The definition obviously is up to some point arbitrary. I have decided to base it on congruence properties of the ambiguity connective $\|$:

Definition

A logic $\vdash_L$ is a logic of ambiguity is logic such that

- $\vdash_L$ *conservatively extends classical logic*
- *L satisfies universal distribution and unambiguous entailments for congruence :*
  1. $\alpha \wedge (\beta\|\gamma) \equiv (\alpha \wedge \beta)\|(\alpha \wedge \gamma)$
  2. $\alpha \vee (\beta\|\gamma) \equiv (\alpha \vee \beta)\|(\alpha \vee \gamma)$
  3. $\neg(\alpha\|\beta) \equiv \neg\alpha\|\neg\beta$
  4. $\alpha \wedge \beta \leqq \alpha\|\beta \leqq \alpha \vee \beta$
  5. $\alpha \wedge \beta \leqq \beta \leqq \alpha \vee \beta$
- *L satisfies idempotence:* $\alpha\|\alpha \equiv \alpha$
- *L satisfies associativity:* $(\alpha\|\beta)\|\gamma \equiv \alpha\|(\beta\|\gamma)$

Meet the family

Note that this definition is very liberal regarding other connectives:

we can have ambiguity logics which do not satisfy

- The DeMorgan laws: $\neg(\alpha \wedge \beta) \equiv (\neg\alpha) \vee (\neg\beta)$
- Distributive laws: $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
- ...

yet they all *extend* classical logic!

Ambiguity logics

The following are straightforward:

Lemma
DAL,$\overline{c}$DAL, TAL,cTAL are ambiguity logics.

Note that the former two are distrustful, the latter two are trustful!

Regarding the notion of trust, we have some nice results:

Outline

The trust lemmas

The intuitive difference between trust (all arguments are schemes) and distrust (arguments are not schemes) is quite clear.

The following should also intuitively hold:

Intuition on trust/distrust

In a setting of trust, more arguments/inferences are valid than in a setting of distrust. The following is a case in point.

(44) a. `Peter likes plants. If someone likes plants, he likes animals.`
     b. $\therefore$ `Peter likes animals.`

How does this intuition with mathematical results?

The trust lemmas

How does this intuition with mathematical results?

Note that obviously not every distrustful ambiguity logic is contained in every trustful ambiguity logic:

- Assume L_d is distrustful, yet commutative, that is, $\alpha\|\beta \vdash \beta\|\alpha$.
- Assume L_t is trustful, yet non-commutative, that is, $\alpha\|\beta \nvdash \beta\|\alpha$.

A general inclusion is not even desirable.

The trust lemmas

However, there are the following strong results:

Lemma

(Trust Lemma 1) Every distrustful ambiguity logic is included in a unique smallest non-trivial a trustful ambiguity logic (its **trust closure***).*

Proof: close under u-substitution.

Lemma

(Trust Lemma 2) No trustful ambiguity logic is contained in a distrustful ambiguity logic.

Proof: closure under e-substitution *preserves* closure under u-substitution, hence would result in triviality!

Hence this asymmetry between trust/distrust can be traced back to different closure properties!

Trust lemmas

Lemma

(Trust lemma 3) Assume L is trustful ambiguity logic such that its inner logic extends CL. Then its inner logic $\leq_L$ is a distrustful ambiguity logic.

These lemmas show that the family, separated into trustful and distrustful members, nonetheless has tight connections between these groups!

Outline

Some more logics

Recall the (alternative) definition of DAL:

$$\alpha \Rightarrow_{DAL} \beta \text{ iff } \blacklozenge\alpha \vdash_{CL} \blacklozenge\beta \text{ and } \blacksquare\alpha \vdash_{CL} \blacksquare\beta$$

We can use the operators $\blacklozenge, \blacksquare$ to construct further logics (procedure by Deemter, [2]):

- $L_{\blacksquare\blacksquare}$: $\alpha \Rightarrow_{\blacksquare\blacksquare} \beta$ iff $\blacksquare\alpha \vdash_{CL} \blacksquare\beta$
- $L_{\blacklozenge\blacklozenge}$: $\alpha \Rightarrow_{\blacklozenge\blacklozenge} \beta$ iff $\blacklozenge\alpha \vdash_{CL} \blacklozenge\beta$
- $L_{\blacksquare\blacklozenge}$: $\alpha \Rightarrow_{\blacksquare\blacklozenge} \beta$ iff $\blacksquare\alpha \vdash_{CL} \blacklozenge\beta$
- $L_{\blacklozenge\blacksquare}$: $\alpha \Rightarrow_{\blacklozenge\blacksquare} \beta$ iff $\blacksquare\alpha \vdash_{CL} \blacksquare\beta$

Some more logics: Construction with truth operators

Lemma

$L_{\blacksquare\blacksquare}, L_{\blacklozenge\blacklozenge}, L_{\blacklozenge\blacksquare}, L_{\blacksquare\blacklozenge}$ are all ambiguity logics.

The reason is that the DeMorgan laws mimic universal distribution.

Lemma

$DAL = L_{\blacksquare\blacksquare} \cap L_{\blacklozenge\blacklozenge}$

This is obvious. Now two surprising results:

Lemma

$L_{\blacklozenge\blacksquare}$ is the smallest ambiguity logic. All inferences valid in this logic are valid in all ambiguity logics.

Some more logics: Construction with truth operators

Lemma

$L_{\blacklozenge\blacksquare}$ is the **smallest** *ambiguity logic. All inferences valid in this logic are valid in all ambiguity logics.*

However, the logic is unreasonably distriustful: it does not even satisfy monotonicity, that is, for example,

$$p \| q \nvdash_{\blacklozenge\blacksquare} p \| q \tag{45}$$

Some more logics: Construction with truth operators

The next is even more surprising:

Lemma
$L_{\blacksquare\blacklozenge} = cTAL$.

So the two logics coincide!

- Note that $L_{\blacksquare\blacklozenge}$ is also quite unreasonable: it allows for ambiguous weakening. Hence it is a logic of "innocence":

$$a \vdash_{CL} b \implies \alpha||a||\beta \vdash_{\blacksquare\blacklozenge} \gamma||b||\delta \tag{46}$$

for arbitrary $\alpha, \beta, \gamma, \delta$.

For most practical reasons, this can be ruled out.

Intermediate summary family

We have seen several methods to construct ambiguity logics:

1. Truth operators
2. Proof-theory
3. Trust closure
4. As inner logics
5. (not treated:) Via algebraic congruences

Many are unreasonable in that they are asymmetric (see $L_{\blacksquare\blacksquare}$) or do not satisfy basic logical equivalences.

However, many are also reasonable – in reasoning with ambiguity, logical pluralism is inevitable!

Intermediate summary family

In particular, different logics are apt for different contexts and purposes:

- Whether we cooperate or we do not (or our partner)
- In linguistics, distrustfulness seems problematic, since lexical meanings always remain somewhat opaque –

  hence we have to reason schematically with them!

Outline

Summary

We have taken a quickie tour through the fabulous worlds of ambiguity logics.

The most important result for this world is the **Fundamental Theorem**: one central closure property must be lacking in all ambiguity logics!

At first glance (and for a long time), I thought of this result as a paradox –

now I think of it rather as the guardian of this world, which preserves it from triviality.

The most noteworthy thing of this world is: seemingly innocuous assumption quickly lead to untenable consequences (see the matter of reducibility)

Purpose

The most faq is: what purpose does all this have? As I have said, ambiguity logics are interesting from four perspectives:

1. Linguistics
2. Cognitive science
3. Philosophy
4. Computer science

To put ambiguity logics to use, we probably need

- Extension to predicate logic (often straightforward)
- A scheme which checks validity of inference under various logics (trustful, distrustful), with fallback options etc.

As van Deemter ([2]) has already put it: the notion of ambiguous consequence is itself ambiguous. We have to deal with this!

Outlook

Here a list with things to be done (which does not mean they have not been yet addressed):

1. We have not yet found a reasonable trustful ambiguity logic which is commutative. Is non-commutativity an inherent feature of trust (i.e. trust comes at the price of plausibility ordering)?
2. The issue of reducibility: all reasonable ambiguity logics so far have irreducibles, yet they are counterintuitive.
3. Are there (meaningful) ambiguity logics without e- and u-substitution? Examples are easy to construct, but those do not make any sense. Can this be generalized?

Outlook

Here a list with things to be done (which does not mean they have not been yet addressed):

4. Which logic is most adequate? Of course, we can deduce many things from abstract properties, but at some point case studies are in order (see Frost-Arnold, Beebe, [4])

Thank you!

- Arapinis, A., Vieu, L.: A plea for complex categories in ontologies. Applied Ontology 10, 285–296 (12 2015)
- van Deemter, K., Ipo, V.D.: Towards a logic of ambiguous expressions (1996)
- van Eijck, J., Jaspars, J.: Ambiguity and reasoning (1995)
- Frost-Arnold, G., Beebe, J.: Confused terms in ordinary language. Journal of Logic, Language and Information 29 (06 2020)
- Pinkal, M.: On semantic underspecification. In: Bunt, H., Muskens, R. (eds.) Computing Meaning: Volume 1, pp. 33–55. Springer, Dordrecht (1999)
- Poesio, M.: Semantic ambiguity and perceived ambiguity. In: van Deemter, K., Peters, S. (eds.) Semantic Ambiguity and Underspecification, pp. 159–201. CSLI Publications, Stanford, CA (1994)

Stallard, D.: The logical analysis of lexical ambiguity. In: Proceedings of the 25th Annual Meeting on Association for Computational Linguistics (ACL '87). pp. 179–185 (1987)

Yarowsky, D.: Unsupervised word sense disambiguation rivaling supervised methods. In: Proceedings of the 33rd Annual Meeting on Association for Computational Linguistics. p. 189196. ACL '95, Association for Computational Linguistics, USA (1995), https://doi.org/10.3115/981658.981684

Thank you (again)!